# INCOMPLETE MARKETS

## Volume I: Finite Horizon Economies

## I. Origin of Ideas

A central theme of classical economic theory since the time of Adam Smith is the role of markets as a mechanism for co-ordinating the activities of many distinct individuals, each acting independently in their own self-interest. An elegant synthesis of two hundred years of economic thought was obtained in the 1950's by Arrow, Debreu and McKenzie[1] which has come to be known as *general equilibrium theory* (GE). While the traditional GE model was static, involving a finite number of consumers, firms and commodities, Arrow (1953, Chapter 1) and Debreu (1959) showed how the analysis could be extended to a general setting with time and uncertainty by introducing an event-tree to describe the uncertainty and a structure of markets in which date-event contingent commodities can be traded at an initial date. This model has come to be known as the *Arrow-Debreu model* (AD). In the AD model a huge collection of contingent contracts—one for each good for each possible date-event in the future—is traded at the initial date and thereafter no further trade takes place; agents simply deliver or receive delivery on the contractual commitments made at the initial date.

A fundamental and far-reaching simplification of this market structure was introduced in Arrow's paper (1953, Chapter 1): he showed that any AD equilibrium could be achieved using an alternative and much more realistic sequential system of markets consisting of *Arrow securities* and *spot markets* at each date-event.[2] An Arrow security purchased or sold at date $t$ is a contract promising to deliver one unit of income in one of the contingencies (date-events) which can occur at date $t+1$. If at each date-event there is a complete set of such contracts that can be traded for all contingencies that can occur at the following date, then any Arrow-Debreu equilibrium allocation can achieved by the equivalent market structure in which agents trade, at each date-event, spot contracts calling for the current delivery of each good, and Arrow securities for the delivery of

[1]Arrow (1951), Debreu (1952, 1959), Arrow-Debreu (1954), McKenzie (1954, 1955, 1959).

[2]Strictly speaking Arrow's paper presents a two-period model: the extension of Arrow's analysis to the multiperiod setting is given by Guesnerie-Jaffray (1974). It is convenient for the discussion here to express the model in the general multiperiod setting.

income at each of the contingencies at the following date. In this way a model of pristine elegance and simplicity is introduced which shows the fundamental role of financial securities when combined with spot markets for achieving an optimal allocation of resources. The highly unrealistic structure of Arrow-Debreu contracts with its concomitant concept of once-and-for-all trade at the initial date is replaced by the much more realistic structure of sequential trade on a system of spot and financial markets.

Arrow's model was the by-product of a felicitous fusion of two strands of economic literature: the Ramsey, von Neumann-Morgenstern and Savage[3] theory of choice under uncertainty and the traditional model of general equilibrium. The then recent development of choice theory had shown how uncertain consumption streams could be formalized as consumption indexed by states of nature and that, with a few hypotheses, an agent's preference ordering over such consumption streams could be represented as an expected utility of the possible outcomes. Introducing a state space for modeling possibles outcomes, expected utility for representing preferences,[4] and combining this with the standard framework of GE gives the Arrow-Debreu model with contingent commodities. This was a major step forward for modeling equilibrium under uncertainty. But perhaps the most innovative contribution of Arrow's paper was its explicit recognition of the problem of optimal risk sharing, and the role that financial markets play in inducing an optimal allocation of risks. This was a true conceptual innovation: for what is so striking from the perspective of today, with all the focus on financial markets and financial innovation, is the conspicuous absence in the preceeding two hundred years of classical economic literature of the need for a theory of risk sharing and its relation to financial markets.[5]

The greater realism of a market structure consisting of spot contracts for goods combined with

[3]Ramsey (1926), von Neumann-Morgenstern (1944), Savage (1954).

[4]Although in the economic literature the state-space approach has its origin in the development of expected-utility theory, the assumption that agents' preferences have an expected-utility representation is not needed in the Arrow-Debreu model. It was however convenient for the sequential model introduced by Arrow, to express optimality of the agents' commodity trades on the spot markets once the uncertainty (at each date-event) is resolved.

[5]The classical economists understood the role of insurance as a mechanism for reducing risks, and the need to compensate agents involved in risky undertakings with a risk premium: see for instance the discussions of Adam Smith (1776) on insurance and on the higher wages of construction workers to compensate for the inevitable fluctuations in their employment (Volume 1, Part 1, Chapter 10). The need for a systematic treatment of risk sharing was however absent, and in the settings where the modern literature sees a trade-off between risk-sharing and incentives, for example in joint-stock ownership or in the metayer system for farming, neither Adam Smith (1776) nor John Stuart Mill (1848) ever mentioned any advantage of such a contractual arrangement for risk sharing, but both focused almost exclusively on the negative effects on incentives.

financial contracts (albeit the idealized Arrow securities) for income transfer is undeniable. But if the equilibrium allocation of such a structure is to coincide with the allocation that would be obtained using the complete, contingent, Arrow-Debreu markets with all trades executed once and for all at an initial date, agents must *correctly anticipate* at the initial date the spot price of every good at every date-event in the future. This is needed in order that the income that agents choose to bring forward at each date-event by their holdings of Arrow securities, permit them to buy the bundle of goods that they had planned to consume when choosing their income transfers. To obtain such a well co-ordinated outcome, agents must have familiarity with the functioning of the economy, and some stationarity in the structure of the economy must prevail in order that agents can be expected to form such correct anticipations.

In a constantly changing, non-stationary world where future events are hard or impossible to foresee, it is surely unreasonable to expect agents to have such perfect foresight: this was the image of the world projected by Keynes in the *General Theory* (1936) and related writings (Keynes (1937)), a vision probably influenced by the turmoil surrounding the First World War and the ensuing Great Depression. The lasting influence of the *General Theory*, and the desire to formalize its ideas in an equilibrium setting, led some researchers in the early seventies to remove the assumption of perfect foresight from the sequential model and to study the outcomes of models in which the agents have exogenous expectations about future spot prices (Grandmont (1970), (1977)) and, if the securities are more general than Arrow securities, the future payoffs of the securities (Green (1972), Hart (1974)). Since agents' expectations of future spot prices are given exogenously, anticipated spot prices cannot be required to be equilibrium prices, or to coincide with the spot prices when the future date-event occurs, and only equilibrium on the current spot and financial markets is required. The economy is viewed as passing through a sequence of "temporary equilibria".

Temporary equilibrium was perhaps too radical a departure from the Pareto optimal world of the Arrow-Debreu theory: its main finding was that existence of a temporary equilibrium only requires that agents have minimal agreement on the future spot prices, more precisely that the supports of their expectations have a non-empty intersection. This is a weak restriction on expectations and, if agents can have almost any expectations about future prices or payoffs, almost any current prices for securities promising delivery in the future can be justified.

Although the temporary equilibrium approach seemed promising as a framework for formalizing

Keynes' ideas, it has proved difficult to work with. While there are assumptions on the formation of expectations which make them less arbitrary, such as adaptive expectations in the multiperiod model, much of the general equilibrium literature, and in particular the part that we are surveying, came back to the approach of perfect foresight, and explored an alternative approach to introducing greater realism into the sequential model. After all Arrow's result on the Pareto optimality of a system of spot and financial markets, and hence the optimal risk sharing achieved by financial markets, hinged on two assumptions: the implicit assumption of perfect foresight and the explicit assumption that for each date-event there is a security which is perfectly tailored to transfer income to this contingency (the Arrow security). It is by recognizing that the available instruments for risk sharing may be incomplete that the theory of *Incomplete Markets* has sought to explore how market imperfections can affect the allocation of resources in an environment of uncertainty.

In the late-sixties-early-seventies two papers appeared which were the first to explore equilibrium with "incomplete" financial markets. Diamond's model (1967, Chapter 2) inspired by the finance literature of the 1960's , focused directly on the "realistic" market structure consisting of the stock and bond markets, while Radner's model (1972, Chapter 3) was inspired by the abstract tradition of AD theory and explored the consequence of replacing Arrow securities by an incomplete set of contracts for the contingent delivery of commodities. Both papers took the financial structure as given, without discussing why the markets might be incomplete. Indeed Radner made it explicit at the end of his introduction that he took the available securities as given "without any explanation of why some contracts are allowed and others are not".

The assumption that there is an exogenously given financial structure is characteristic of the incomplete markets literature, and of most of the papers collected in this volume.[6] While the structure of the securities embedded in the model is progressively enriched, the reasons for the incompleteness are typically left out of the discussion and are not present in the model. This is both the strength—the structure of the model is sufficiently close to that of the AD model to make it amenable to the powerful techniques of analysis developed for that framework—and the weakness of the approach: since the reasons for incompleteness are not modeled, one may wonder if a more general model which takes into account the reasons for incompleteness may not lead to somewhat different conclusions. Before entering into a discussion of the properties of incomplete

[6] The exceptions are the papers in Part VI.

market economies described in the essays of this volume, it may therefore be useful to pause for a moment to discuss what is "hidden" behind the assumption that markets are incomplete.

In the introduction to the book *Theory of Incomplete Markets* (Magill-Quinzii (1996)) we have attempted to provide a conceptual framework for understanding why the structure of markets that we observe is not like that postulated in the Arrow-Debreu model, but rather consists of sequential trades on a system of spot and financial markets with only limited contracts for meeting future uncertain contingencies. The arguments are an adaptation of those proposed by Williamson (1985) for showing more generally how different customs, institutions and market structures evolve in a historical context. The key hypothesis is that agents are *boundedly rational*—they have limited time, ability and resources to process all the information that would be required for fully rational behavior—and they are *opportunistic*—agents acting in their own self interest will, whenever possible, renege on their contracts if it is in their interest to do so, and exploit to their advantage the non-observability of their actions or characteristics. In such a world, since making all the necessary calculations for trade in all future contingencies, for all commodities, for all future time, is costly, and since monitoring and enforcing all such contracts involves additional costs, the overall cost of running a system based on Arrow-Debreu contracts would be prohibitive.

A division of labor that uses spot markets for allocating commodities and financial markets for allocating income leads to a system which is much less costly to operate. For the simplest of all contracts are those for which the date of issue and the date of maturity coincide: using such spot contracts to allocate the commodities minimizes on the need to plan into the future, and avoids the problem of reneging on delivery. However, if these were the sole markets on which agents could trade, then their consumption would always be tied to their current income: financial markets enable agents to redistribute their income over time and across future contingencies, but the extent to which such markets can be used is inevitably limited by the bounded rationality of participants, and by opportunistic behavior.

The bounds on rationality are continually evolving. The expanding use of computers providing enhanced computing power and the almost effortless dissemination of information through the World Wide Web, have greatly reduced the transaction costs of using financial markets. This decrease in costs accounts for the rapid pace of financial innovation in the last thirty years, during which agents grew accustomed to trading instruments of progressively greater complexity, such as the array of derivative securities used by financial institutions, and the sophisticated mortgage contracts offered to consumers.

If the bounds on rationality evolve with time, making more complex contracts tradable, opportunism is a basic attribute of human behavior which will not change and will always set limits to the possibilities for risk sharing. Except for spot contracts, every contract involves an exchange in which the two sides of the exchange take place at different points in time. One agent makes payment (delivery) today in exchange for a promise by the other agent to make a contingent delivery in the future. And herein lies the rub: for opportunism implies that the agent with the obligation to make delivery will, whenever possible, renege on the commitment or claim that the contingency calling for delivery has not occurred. Thus contracts involve costs of monitoring, verification, and enforcement, and explicit or implicit penalties for reneging on commitments: only those contracts can survive for which the benefits from the exchange outweigh the costs incurred in their enforcement.

Radner's (1972, Chapter 3) paper has provided the basic model of general equilibrium with incomplete markets. At the time it was written, contract theory and economics of information were in their infancy, and a theoretical framework for analyzing optimal contracts in the presence of moral hazard and adverse selection had not yet appeared. The assumption that agents in the economy have access to a limited collection of financial contracts for meeting future contingencies was viewed as a convenient and natural short-cut for constructing a model to study the consequences of imperfect markets for risk sharing.

## II. Existence of Equilibrium for Exchange Economies

Radner's paper was concerned with proving existence of what we may call *a spot-financial market equilibrium.* Under standard assumptions of convexity and continuity of agents' preferences and technology sets of firms—and the additional assumption that agents' trades are bounded—Radner proved the existence of equilibrium for an exchange economy; the analysis was less satisfactory for a production economy because of the somewhat arbitrary choice of objective functions for firms (a topic we shall return to later). As will be abundantly clear from this volume, the *exchange* version of the model has received much more attention than the model with *production*—a state of affairs which seems regrettable, since the way firms cope with uncertainty in their investment decisions is fundamental for the long-run growth of the economy. There is perhaps a good reason for this since, as we shall see, the model with production is very hard to treat in a satisfactory way. A more positive view of the emphasis on the exchange model is that it has served to bring general equilibrium close to finance, GE providing a theoretical foundation for financial economics.

In an influential paper Hart (1975, Chapter 5) focused on a more detailed analysis of the exchange version of Radner's model. He showed that this new class of sequential models of equilibrium with incomplete markets had properties that were new to equilibrium theory. For a start, even in an economy in which agents have standard convex continuous preferences, the sequential structure of the model and the fact that some securities can have payoffs which vary with the spot prices of the commodities that they promise to deliver, implies that without artificial bounds on trades an equilibrium may not exist. Furthermore the markets can be led to misallocate resources in the sense that an equilibrium may fail to be *constrained optimal* (a concept that we will explain shortly). Finally Hart laid to rest the conventional wisdom that more markets lead to better outcomes, by showing a case in which adding one security leads to an equilibrium in which all agents are worse off. The discovery of these somewhat surprising properties of the sequential model came as a shock to economic theorists, and presented them with a challenge that has taken more than ten years to resolve. Many of the papers in this volume are either a direct attempt to provide a response to a problem raised in Hart's paper, or belong to a strand of research that it directly inspired.

The papers in Part II provide the tools and analytical results required to establish the (generic) existence of an equilibrium. To discuss the content of these papers it will be convenient to introduce some of the basic notation commonly used in this branch of general equilibrium.

Consider the simplest two-period ($t = 0, 1$) exchange economy with $L$ commodities and $I$ agents, where each agent is uncertain about his endowment of the goods at date 1. We assume that uncertainty is expressed by the fact that "nature" will draw one of $S$ possible "states of nature", say $s \in \{1, \dots, S\}$, and though an agent does not know which state will be chosen, he does know what his endowment $\omega_s^i = (\omega_{s1}^i, \dots, \omega_{sL}^i)$ will be if state $s$ occurs. For convenience label date 0 as state 0: then agent $i$'s endowment is $\omega^i = (\omega_0^i, \omega_1^i, \dots, \omega_S^i)$, $i = 1, \dots, I$. Agents can exchange goods and share their risks by trading on spot markets (one for each good in each state) and a collection of security markets. Let $p_{sl}$ denote the spot price of good $l$ in state $s$, and let $p_s = (p_{s1}, \dots, p_{sL})$ denote the vector of spot prices in state $s$; then $p = (p_0, \dots, p_S)$ denotes the vector of spot prices across all date-events in this two-period setting. A similar notation is used for allocations.

At date 0 there are also $J$ securities ($j = 1, \ldots, J$) which agents can trade. Security $j$ is a promise to pay $V_s^j$ if state $s$ occurs, where the payment $V_s^j$ is measured in the unit of account of state $s$, $s = 1, \ldots, S$. We say that security $j$ is *real* if it is a promise to deliver the value of a bundle of goods $A_s^j = (A_{s1}^j, \ldots, A_{sL}^j)$ in each state $s$ so that $V_s^j = p_s A_s^j$. Security $j$ is said to be *nominal* if its payoff $V_s^j$ is independent of the spot prices $p_s$. Whatever the type of the security, its price at date 0 is denoted by $q_j$, and the vector of all security prices is $q = (q_1, \ldots, q_J)$.

Each agent trades on the financial markets choosing a portfolio $z^i = (z_1^i, \ldots, z_J^i)$ of the securities. These transactions on the financial markets redistribute the agent's income across time and the states. The income acquired or sacrificed at date 0 is $-qz^i = -\sum_{j=1}^J q_j z_j^i$ (if $z_j^i < 0$, agent $i$ sells security $j$, or equivalently uses security $j$ to borrow; if $z_j^i > 0$, agent $i$ buys security $j$, or uses security $j$ to save). The income earned or due in state $s$ is $V_s z^i = \sum_{j=1}^J V_s^j z_j^i$, where $V_s$ denotes row $s$ of the $S \times J$ matrix $V$ of security payoffs. These income transfers serve to finance the excess expenditures $p_s(x_s^i - \omega_s^i)$ of the planned consumption stream $x^i = (x_0^i, x_1^i \ldots, x_S^i)$. Thus the agent's budget set, when current and anticipated prices are $(p, q)$, is given by

$$\mathcal{B}(p, q, \omega^i) = \left\{ x^i \in \mathbb{R}_+^{L(S+1)} \;\middle|\; \begin{array}{lcll} p_0(x_0^i - \omega_0^i) & = & -qz^i, & z^i \in \mathbb{R}^J \\ p_s(x_s^i - \omega_s^i) & = & V_s z^i, & s = 1, \ldots, S \end{array} \right\} \tag{1}$$

Each agent $i$ has a preference ordering over the consumption streams $x^i \in \mathbb{R}_+^{L(S+1)}$ which is represented by a utility function $u^i : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$ which is typically assumed to have "nice" properties of strict quasi-concavity, monotonicity, and smoothness.

An equilibrium of "plans, prices, and price expectations" in Radner's terminology, also called a *spot-financial market equilibrium* or a *GEI equilibrium* (general equilibrium with incomplete markets), is defined as a pair of actions and prices $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ such that

(i) $(\bar{x}^i, \bar{z}^i)$ maximizes $u^i(x^i)$ over the budget set $\mathcal{B}(\bar{p}, \bar{q}, \omega^i)$, $i = 1, \ldots, I$.

(ii) the spot markets clear: $\sum_{i=1}^I (\bar{x}_s^i - \omega_s^i) = 0, \quad s = 0, \ldots, S$.

(iii) the financial markets clear: $\sum_{i=1}^I \bar{z}^i = 0$.

The market-clearing conditions (ii) for the agents' planned consumption vectors at date 1 (for $s = (1, \ldots, S)$) is what Radner called an equilibrium of "plans" since the planned consumptions of all agents are compatible, and the anticipated vector of prices $p_s$ for each states $s$ will be equilibrium vector of spot prices if state $s$ occurs (equilibrium of "expectations").

Radner's approach to proving that a GEI equilibrium exists was to restrict agents' financial trades, i.e. he imposed a constraint of the type $z_j^i \leq \alpha$ for some $\alpha > 0$. Hart gave a simple example of an economy in which, if there is no bound on portfolios, no equilibrium exists and, if there is a bound, agents trade up to this arbitrary given bound. The example turns out to have a natural economic interpretation (see Magill-Shafer (1991, p.1538)). It is constructed in an economy with two goods and no aggregate risk, in which the (real) securities are futures contracts on the goods. In such an economy if the agents are subject to individual risks and have distinct preferences for the goods, a risk sharing equilibrium cannot be obtained by using futures contracts. Since there is no aggregate risk, when agents trade on the futures markets they reduce the variability of future spot prices, but this destroys the ability of futures contracts to share their risks—and in the limit when the spot prices cease to fluctuate, the payoffs of the futures contracts become collinear and the system breaks down. Hart's example is clever and carries a broader message: financial securities need to be adapted to the kind of risks they are supposed to help agents share: in an economy without aggregate risk, futures contracts, which "live" on price variability, are not the appropriate risk-sharing instruments. Imposing bounds, while apparently solving the problem of existence, in fact hides a deeper problem, the potential incompatibility between the operation of the spot and financial markets for some risk profiles $(\omega^1, \ldots, \omega^I)$ and security structures $V$. Furthermore, even though bounds may reflect realistic restrictions (for example margin requirements and the like) imposed on typical consumers, they are not so obviously realistic if the model is used to study trading on financial markets by professionals who can and do take large offsetting positions on different contracts. For these reasons the subsequent literature has chosen to study the GEI model without imposing bounds on agents' trades. The focus has been on understanding whether the incompatibility between risk and security structure $(\omega^1, \ldots, \omega^I, V)$ exhibited by Hart is a pervasive or an exceptional phenomenon.

The first papers to develop a general method for dealing with models without bounds on trades were Cass (2006, Chapter 6) and Magill-Shafer (1990, Chapter 7) (MS for short). These papers which were written independently and contemporaneously around 1984-1985, were published much later and, together with the paper of Geanakoplos-Polemarchakis (1986, Chapter 14), marked the beginning of an attempt to formulate a general approach to the study of economies with incomplete markets. The key idea behind the approach adopted by Cass and MS can be motivated as follows. A GEI equilibrium as defined above, while natural as an economic concept, is not well behaved as a mathematical object. This is because the portfolio trades $(z^i)$ are just an indirect means to an end—to redistribute income across the states so that the desired consumption stream $x^i$ can be purchased. What is needed is a way of expressing a GEI equilibrium which retains the symmetry (duality) between consumption streams and prices present in the Arrow-Debreu formulation, while incorporating the requisite constraints on the transfers of income.

The $S+1$ equations in the budget set $\mathcal{B}(p, q, \omega^i)$ in (1) simply state that the excess expenditures on the spot markets $p_0(x_0^i - \omega_0^i)$ at date 0 and $p_s(x_s^i - \omega_s^i)$ in each state $s$ at date 1 must be financed with the income obtained from the financial markets $(-qz^i, V_s\, z^i,\ s = 1, \ldots, S)$. Clearly if the price vector $q$ is such that a portfolio $z^i$ can be found which makes this latter expression "semi-positive" (each component is non-negative and one is strictly positive) then there will be no solution to the maximum problem of agent $i$, for he will attempt to use this portfolio on an arbitrarily large scale. Thus the price vector $q$ must prevent the existence of such "arbitrage opportunities". This simple observation has important consequences. Basically it asserts that it must not be possible to acquire income for free in any state or, more precisely, it can be shown that it implies that there is a vector of positive prices $\pi = (\pi_1, \ldots, \pi_S)$ such that

$$q_j = \sum_{s=1}^{S} \pi_s V_s^j, \quad j = 1, \ldots, J \Longleftrightarrow q = \sum_{s=1}^{S} \pi_s V_s \tag{2}$$

$\pi_s$ is the present value at date 0 of one unit of income in state $s$, so that (2) asserts that the price $q_j$ of security $j$ is the present value of its date 1 dividends $V_s^j$. (2) implies that for any trade $z^i$

$$-qz^i + \sum_{s=1}^{S} \pi_s V_s^j z^i = 0 \tag{3}$$

so that trading on the financial markets does not add to the present value of agent $i$'s income.

The present-value prices for income implicit in the security prices can be used to define present-value prices for all the goods: let $P_{0l} = p_{0l},\ P_{sl} = \pi_s p_{sl}, s = 1, \ldots, S,\ l = 1, \ldots, L$. $P_s$ is the vector of present-value prices of the goods in state $s$, i.e. $P_{sl}$ would be the price at date 0 of the promise to deliver one unit of good $l$ in state $s$ if such (Arrow-Debreu) contracts were traded. Replacing in (3) the income coming from financial trades by the planned expenditures that they finance gives the present-value budget constraint

$$P(x^i - \omega^i) \equiv \sum_{s=0}^{S} P_s(x_s^i - \omega_s^i) = 0 \tag{4}$$

Since trading on financial markets does not alter the present value of an agent's income, each agent must restrict the choice of consumption streams to those whose present-value cost does not exceed the present value of the agent's income.

In addition to this constraint, the date 1 expenditures must be financed by trading on the possibly limited set of securities (the date 1 budget constraints in (1)). If we write $P = (P_0, P_{\mathbf{1}})$ where $P_{\mathbf{1}} = (P_1, \ldots, P_S)$ is the date 1 vector of present-value prices, then this condition can be expressed in geometric form as

$$P_{\mathbf{1}} \circ (x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) \in \langle V(P_{\mathbf{1}}) \rangle \tag{5}$$

where $P_{\mathbf{1}} \circ (x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) = (P_1(x_1^i - \omega_1^i), \ldots, P_S(x_S^i - \omega_S^i))$ is the vector of date 1 excess expenditures and $\langle V(P_{\mathbf{1}}) \rangle$ is the subspace spanned by the columns of the date 1 payoffs of the $J$ securities. Thus when the asset price $q$ prevents arbitrage and satisfies the price relation (2), the budget set $\mathcal{B}(p, q, \omega^i)$ can be expressed equivalently in terms of the present-value prices $P$ by the two conditions (4) and (5). Going from (1) to (5) requires multiplying each date 1 budget constraint by the corresponding state price $\pi_s$: this does not change the equation when the securities are real ($V_s^j = p_s A_s^j$), as studied by MS. When the securities are nominal, Cass (2006, Chapter 6) exploits the indeterminacy of the price level in each state in the standard definition of a GEI equilibrium to justify equation (5) (with $\langle V(P_{\mathbf{1}}) \rangle = \langle V \rangle$).

Expressing the budget equations of an agent by (4) and (5) transforms the GEI equilibrium into a more symmetric form —which has proved to be a powerful and convenient tool—since the financial variables no longer appear explicitly, and the market-clearing conditions can be expressed as standard market-clearing equations for goods. The last step in transforming a GEI equilibrium, which brings it as close as possible to the Arrow-Debreu (GE) formulation, exploits the fact that when $I-1$ agents satisfy the date 1 equations (5), and the commodity markets clear, then necessarily the remaining agent satisfies these constraints. *Thus the budget set of one agent, say agent 1, can be reduced to the constraint (4), omitting constraint (5).*[7] This property is crucial for establishing

[7] Omitting the subspace constraint (5) for one agent was referred to by Geanakoplos (1990) as the "Cass trick", a terminology often used since. This construction was first used by Cass (Chapter 6, CARESS Discussion Paper 1984)

the boundary behavior of aggregate excess demand when some of the present-value prices tend to zero.

At this point the proof of existence is essentially done for the case of nominal assets studied by Cass since the subspace $\langle V \rangle$ is fixed, the demand of each constrained agent (i.e an agent with constraints (4) and (5)) is continuous, and the demand of the unconstrained agent (the agent without constraint (5)) is continuous and gives the required boundary behavior for the aggregate demand. An alternative proof for this case is given in Geanakoplos-Polemarchakis (1986, Chapter 14).

When the securities are real, as the prices $P_{\mathbf{1}}$ of the date 1 goods change, the subspace $\langle V(P_{\mathbf{1}}) \rangle$ itself changes and can drop in dimension, causing discontinuities in agents' demand functions, which can lead to non-existence of an equilibrium as in Hart's example. MS provided a framework for systematically studying these drops in rank in models with real securities. They studied the case where there are $J = S$ real securities and the bundles $(A^j)_{j=1}^J$ which serve as the basis for the payoffs of the securities are such that, for "most" date 1 prices $P_{\mathbf{1}}$, the rank of $V(P_{\mathbf{1}})$ is $S$. When the rank is maximal, (5) does not introduce any additional constraint since $\langle V(P_{\mathbf{1}}) \rangle = \mathbb{R}^S$, and agents only have the present-value budget constraint (4) as in an Arrow-Debreu equilibrium.

MS showed the following properties: if the economies are parametrized by the agents' endowments $\omega = (\omega^1, \ldots, \omega^I)$, then for "almost all" endowments (i) there is an equilibrium in which the payoff matrix $\langle V(P_{\mathbf{1}}) \rangle$ has full rank and (ii) there are no low rank equilibria. Establishing these properties required the use of techniques of differential topology, which permit statements like a "property is true for almost all parameter values " to be formalized and proved. These techniques had been introduced by Debreu (1970) for the GE model and had already established themselves as a powerful framework for analyzing qualitative properties (local uniqueness and comparative statics) of the standard GE model. As can be seen from the papers in this volume, they subsequently proved to be essential tools for analyzing the GEI model. Establishing the property that for almost all endowments there are no low-rank equilibria required a new way of analyzing equilibria, giving agents a subspace $H$ of dimension $\rho < S$ for their income transfers in (5), and requiring that at equilibrium $\langle V(P_{\mathbf{1}}) \rangle = H$ (expressed as a system of equations).

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and used independently by Magill-Shafer (Chapter 7, MRG Discussion Paper 1985) to prove generic equivalence between Arrow-Debreu and Financial Markets equilibrium when there are as many securities as states of nature.

The important step of proving (generic) existence of equilibrium with real securities and incomplete markets was taken by Duffie-Shafer (1985, Chapter 8) (DS for short) who showed that if the date 1 constraints (5)—which create the potential changes of rank and discontinuities— are replaced by the conditions

$$P_{\mathbf{1}} \circ (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in H, \qquad \langle V(P_{\mathbf{1}})\rangle \subset H, \qquad \text{for } i = 2, \ldots, I$$

where $H$ is a subspace of $\mathbb{R}^S$ of dimension $J$, then a modified "constrained Arrow-Debreu equilibrium" —which they called a *pseudo-equilibrium*—is obtained, which no longer has discontinuities since the subspaces of income transfers always have the same dimension. Such an equilibrium is defined by a system of equations

$$\begin{cases} F(P, H, \omega) & = & 0 \\ \langle V(P_{\mathbf{1}})\rangle & \subset & H \end{cases} \qquad \Longleftrightarrow \qquad G(P, H, \omega, A) = 0 \tag{6}$$

where $F$ is the aggregate excess demand for each commodity, the prices $P$ lie in the simplex in $\mathbb{R}^{L(S+1)}_{+}$, the subspace $H$ lies in the set of all subspaces of dimension $J$ in $\mathbb{R}^S$, called the *Grassmanian* $G^{J,S}$, the agents' endowments $\omega$ lie in $\mathbb{R}^{L(S+1)I}_{++}$, and the asset commodity payoffs $A$ (the goods in terms of which the value $V$ of their payoffs are based) lie in $\mathbb{R}^{SLJ}$. Since a subspace of dimension $J$ in $\mathbb{R}^S$ can be represented by a system of $S - J$ linear equations, the inclusion $\langle V(P_{\mathbf{1}})\rangle \subset H$ can be expressed by a system of equations, leading to a pseudo-equilibrium of an economy with endowment-asset structure $(\omega, A)$ as the solution of the system of equations $G(P, H, \omega, A) = 0$ in (6). To prove existence of a solution, Duffie-Shafer drew on the method of degree theory introduced by Balasko (1976, 1988) to prove existence of an equilibrium for the GE model.

It might be helpful to explain the idea underlying degree theory. Suppose $f : M \to N$ is a smooth map from a compact manifold $M$ to a (connected) manifold $N$ of the same dimension: heuristically a manifold of dimension $k$ is a hypersurface in an ambient space $\mathbb{R}^m$ which has at each point a linear tangent space of dimension $k$. An element in the range $N$ is called a regular value of the map $f$ if the linear approximation $df_x$ (associated with the matrix of partial derivatives) is of maximal rank, for any $x$ in the preimage of $y$, $x \in f^{-1}(y)$. The basic theorem of degree theory asserts that for all regular values $y \in N$, the mod 2 number of points in the preimage $f^{-1}(y)$ —0 if the number is even, 1 if it is odd—is constant, and is called the *mod 2 degree* of the map $f$. If this number is 1, the preimage $f^{-1}(y)$ contains an odd number of points and therefore is not empty.

To show how this property can be exploited to prove existence of a pseudo-equilibrium, consider the system of equations $G(P, H, \omega, A) = 0$. When properly written, the number of equations is the same as the number of unknowns $(P, H)$ so that the set of solutions $(P, H, \omega, A)$ of these equations

$$\mathbb{E} = \{(P, H, \omega, A) \in S \times G^{J,S} \times \mathbb{R}^n \mid G(P, H, \omega, A) = 0\} \tag{7}$$

is a manifold of the same dimension as the set of endowment-asset payoffs $(\omega, A) \in \mathbb{R}^n$. If the map $f : \mathbb{E} \to \mathbb{R}^n$ which projects $\mathbb{E}$ onto the space of endowment-asset payoff pairs—i.e. $f(P, H, \omega, A) = (\omega, A)$—has a mod 2 degree 1, then for each $(\omega, A)$ there is a corresponding pair $(P, H)$ in the "equilibrium manifold" $\mathbb{E}$, so that a pseudo-equilibrium exists. Using the property of the projection when the endowment $\omega$ is a Pareto optimum, DS showed that the mod 2 degree of $f$ is 1, thus establishing the existence of a pseudo-equilibrium. Arguments similar to those in MS then show that low-rank pseudo-equilibria, where rank $V(P_{\mathbf{1}}) < J$ and the inclusion in (6) is strict, are exceptional, so that for generic economies $(\omega, A)$ a pseudo-equilibrium is a GEI equilibrium.

A proof of existence which exploits degree theory in a more general setting was established virtually contemporaneously with Duffie-Shafer by Husseini-Lasry-Magill (1990) although the paper was published later. A simpler geometric proof of their result based on more intuitive arguments of intersection theory was subsequently given by Hirsch-Magill-Mas-Colell (HMM for short) (1990, Chapter 9). HMM look at the system of equations $G(P, H) = 0$ for fixed $(\omega, A)$ and show that $G$ can be written as

$$G(P, H) = \Big(F(P, H), proj_{H^\perp} V^1(P_{\mathbf{1}}), \ldots, proj_{H^\perp} V^J(P_{\mathbf{1}})\Big)$$

where $F$, the vector of aggregate excess demands for each good, is an inward pointing vector field on the price space—here taken to be the positive unit sphere—and the second component is a section of a vector bundle over the Grassmanian, the so called " orthogonal vector bundle" (see Chapter **??** for definitions). A generalization of mod 2 degree theory outlined above to vector fields and intersections of sections of vector bundles establishes the existence of a solution to the system of equations $G(P, H) = 0$. The geometric reason why every section of the orthogonal vector bundle intersects the "zero section" is rather intriguing—it is linked to the twisting of the bundle which, in the simplest case, reduces to the Möbius band.

The papers on existence of equilibrium with real asset structures are fairly technical. An introduction to the techniques used in these papers and the relation between the results can be found in the Magill-Shafer survey paper (1991). The problem of proving existence of a pseudo-equilibrium received considerable attention in the 1980's: in addition to the DS and HMM approach outlined above, another approach was provided by the paper of Geanakoplos-Shafer (1990), who insert the condition $\langle V(P_{\mathbf{1}})\rangle \subset H$ in the domain of the variables $(P, H)$ on which the excess demand functions are defined, and show, using degree theory, that this system of equations has a solution. Such an approach may well have promising applications for proving existence of equilibrium in other settings—Geanakoplos-Shafer apply their method to prove existence of a marginal cost pricing equilibrium in non-convex economies.

The question of computing equilibria of economies with real securities raises interesting questions which are addressed in the paper of Brown-DeMarzo-Eaves (BDE) (1996). The theoretical approach to proving existence in Chapters 8 and 9 suggest computing pseudo-equilibria, for which there are no discontinuities, since generically these are true GEI equilibria. However BDE found that in practice taking as unknowns the prices and the parameters needed to identify a subspace leads to a high-dimensional problem which quickly becomes difficult to solve numerically. They introduce an ingenious method for computing an equilibrium with only the prices $P$ as unknowns, using the demand of the unconstrained agent to "fill in" when there is a drop of rank of the payoff matrix $V(P_{\mathbf{1}})$.

### III. Production and the Stock Market

It has long been a tradition in economics to decompose the study of resource allocation into two parts: first the problem of production of goods and second the problem of their distribution and exchange. While the classical economists Adam Smith (1776), Mill (1848), and the first formalizer of equilibrium theory, Walras (1874), placed great emphasis on the importance of understanding production and capital accumulation, ever since the modern formalization of the theory of markets by Arrow-Debreu in the 1950's, there has been a tendency for general equilibrium theory to focus on the analysis of exchange economies. Fortunately, macroeconomics, which has moved progressively closer to general equilibrium has reestablished a balance, by focusing attention on the short-run consequences of the business cycle and on the long-run consequences of capital accumulation, albeit from an aggregate perspective. Some of these contributions which are linked to the theory of incomplete markets will appear in Volume 2 when we examine infinite-horizon economies.

The activity of production introduces a new interface with financial markets which arises from a simple fact: production takes time. The cost of investment must be incurred before the revenue is obtained from the sale of the output. This mismatching of disbursements and receipts implies that every production plan must be accompanied by an appropriate method of financing. Furthermore there is a close connection between the method used to finance investment and the ownership structure of a firm.

Three principal types of ownership structures can be distinguished: the *sole proprietorship* (individually-owned firms), the *partnership* and the *corporation.* For the first two types of firms, the entrepreneur or partners finance the investment: they own and control the firm, acting in their own best interest. A corporation sells ownership shares on the stock market: it is owned by shareholders and managed by a separate group of agents specialized in the job of management. The papers in this section focus on the corporation, and in particular on the rules that managers should follow to choose investment in the best interest of the shareholders.

In the standard general equilibrium model with production each firm maximizes profit, taking prices as given, and this leads to a Pareto optimal allocation. Ownership structure does not matter since all owners agree with the objective of profit maximization for each firm. When markets are complete, the sequential model with spot-financial markets and perfect foresight is equivalent to the AD model, and the profit in the AD economy is equivalent to the present value of the profit in the sequential model. For, as we have seen above, the equilibrium prices $q = (q_1, \dots, q_J)$ of the securities must be arbitrage free, and from this we can deduce the existence of present-value prices $\pi = (\pi_1, \dots, \pi_S)$ for income in each state at date 1 such that the price $q_j$ of a security coincides with the present value $\sum_{s=1}^{S} \pi_s V_s^j$ of its payoff stream. When markets are complete—$V$ has $S$ linearly independent payoff streams, which requires $J \geq S$—then the vector of present-value prices $\pi$ which satisfies (2) for a given vector $q$ is unique. In this case the economy is equivalent to an AD economy with present-value prices $P_{sl} = \pi_s p_{sl}$ so that, if the production plan chosen by each firm maximizes the present value of its profit with price vector $P$, then the equilibrium outcome is Pareto optimal.

When markets are incomplete, there are many solutions $\pi$ to the no-arbitrage equations (2), so that the present value of profit is no longer unambiguously defined. However, even with incomplete markets, there are cases for which maximizing the present value of profit is still meaningful and is the appropriate objective for a corporation.

To see this, consider the problem of defining an objective for a corporation in the simplest one-good ($L = 1$) finance economy, the case studied by all the papers in the production section and the two classics, Diamond (1967, Chapter 2) and Drèze (1974, Chapter 4). In this case we can normalize the spot price of the good by setting $p_{s1} = 1$ for all states and ignore the spot markets. A firm's production plan—say firm $k$'s production plan—consists of an input-output pair $y^k = (y_0^k, y_{\mathbf{1}}^k) = (y_0^k, y_1^k, \ldots, y_S^k)$, where $y_0^k$ is the investment made at date 0 and $y_{\mathbf{1}}^k$ is the risky stream of income it generates at date 1. In this setting, Diamond (1967, Chapter 2) studied a production economy with $K$ firms, in which the securities consisted of the equity of the $K$ firms and a riskless bond. He introduced the concept of a *stock market equilibrium* in which firms use the information contained in the stock market prices to maximize the market value of their production plans.

To understand how market-value maximization can be given a well-defined meaning even though the financial markets are incomplete, suppose that any plan $y^k$ feasible for firm $k$ has a date 1 income stream which is a combination of income streams priced by the market: $y_{\mathbf{1}}^k \in \langle V \rangle$. Then a competitive (price taking) firm can use the market prices of the securities to evaluate the market value of its plan $y^k = (y_0^k, y_{\mathbf{1}}^k)$: since $y_{\mathbf{1}}^k = \sum_{j=1}^J \alpha_j^k V^j$ for some coefficients $(\alpha_j^k)$, the market value of the plan must be $\sum_{j=1}^J \alpha_j^k q_j - y_0^j$. Thus with this spanning condition, not only is market-value maximization well defined but it is equivalent to maximizing the present value of profit. For any vector of present-value prices $\pi = (\pi_1, \ldots, \pi_S)$ satisfying the no-arbitrage relation (2), the present value of a plan $y_{\mathbf{1}}^k \in \langle V \rangle$ is equal to

$$\sum_{s=1}^S \pi_s y_s^k - y_0^k = \sum_{s=1}^S \pi_s \left( \sum_{j=1}^J \alpha_j^k V_s^j \right) - y_0^k = \sum_{j=1}^J \alpha_j^k q_j - y_0^k$$

so that maximizing the market value of the firm is equivalent to maximizing the present value of its profit, for any admissible vector of present-value prices.

In Diamond's model when the number of firms is smaller than the number of states of nature the financial markets are incomplete. However the technology of each firm is contained in a one-dimensional subspace, so that once a production plan is valued (by the firm's equity), the condition $y_{\mathbf{1}}^k \in \langle V \rangle$ is automatically satisfied. As a result the value of the equilibrium production plan is sufficient to value any alternative production plan for the firm.

Ekern and Wilson (1974, Chapter 10) and Radner (1974, Chapter 11) subsequently showed that Diamond's condition could be generalized, so that the concept of a stock market equilibrium could be analyzed in a broader setting. The condition, formalized by Radner (1974, Chapter 11), has come to be called *partial spanning* and requires that there is a linear subspace $Z \subset \mathbb{R}^{S+1}$ of dimension $N \leq S$ such that each firm's technology set is a subset of $Z$. An important special case of partial spanning occurs when the firm's technology sets have a factor structure: there are $N$ linearly independent factors $\eta^1, \dots \eta^N$ where $\eta^n$ lies in $\mathbb{R}^S_+$, and concave increasing functions $f^k_n(y^k_0)$ (one for each firm $k$ and each factor $n$) such that the input-output feasible pairs $(y^k_0, y^k_{\mathbf{1}}) \in Y^k$ are such that $y^k_{\mathbf{1}} = \sum_{n=1}^N \eta^n f^k_n(y^k_0)$. Diamond's ray technology sets correspond to the special case where each firm is influenced by only one of the factors.

Another conceptual innovation was introduced in Diamond's paper, which has proved fruitful for the development of the theory of incomplete markets. This is the idea that applying the standard Pareto criterion is not an appropriate procedure for judging the efficiency of markets when they are incomplete, since the incompleteness of financial markets necessarily implies that only a subset of the feasible allocations can be attained. He argued that the appropriate procedure for judging the efficiency of markets consists in first restricting the allocations (the income transfers) to those which can be attained with the existing securities—the so-called *constrained feasible allocations*—and then applying the Pareto criterion to this restricted set. Diamond showed that when firms have ray technology sets, the stock market equilibrium which he defined is constrained Pareto optimal.

Partial spanning is an interesting assumption which guarantees that the condition $y^k_{\mathbf{1}} \in \langle V \rangle$ holds for all firms $k$ and all feasible date 1 production plans $y^k_{\mathbf{1}}$, but it is nevertheless restrictive. Drèze (1974, Chapter 4) was the first to face the problem of defining an objective function for a corporate firm in a setting where partial spanning no longer holds, and thus to extend the concept of a stock market equilibrium with incomplete markets to a general setting. For, if a firm can envision a production plan which lies outside the span of the markets at equilibrium, then the market value of such a plan is no longer well defined, and it is not clear what action the firm should take to serve the best interest of its owners.

To understand Drèze's approach, consider the portfolio problem of a typical agent $i$ in a one-good two-period economy with $J$ securities whose payoffs in the $S$ states at date 1 are summarized by the matrix $V$. The consumer must choose the portfolio of securities $z^i = (z^i_1, \dots, z^i_J)$ to purchase on the financial markets. This choice determines the consumption streams $x_0^i = \omega_0^i - qz^i$ at date 0 and $x_s^i = \omega_s^i + \sum_{j=1}^J V_s^j z^i$ in each state at date 1 (where $\omega^i$ denotes the initial resources of the agent). This is just the simplification of the multigood budget set of the previous section to the one-good case. If $u^i(x^i)$ is the agent's utility function, the agent will choose the portfolio $z^i$ which solves the problem

$$\max_{z^i \in \mathbb{R}^J} u^i(\omega_0^i - qz^i, \omega_1^i + \sum_{j=1}^J V_1^j z^i, \ldots, \omega_S^i + \sum_{j=1}^J V_S^j z^i)$$

for which the first-order conditions are

$$-\frac{\partial u^i}{\partial x_0^i}(x^i) q_j + \sum_{s=1}^S \frac{\partial u^i}{\partial x_s^i}(x^i) V_s^j = 0, \qquad j = 1, \ldots, J$$

namely that the marginal cost of acquiring an additional unit of security $j$ at date 0 should equal the marginal benefit derived from its payoff at date 1. For each state $s$ the number

$$\pi_s^i(x^i) = \frac{\partial u^i}{\partial x_s^i}(x^i) \Big/ \frac{\partial u^i}{\partial x_0^i}(x^i)$$

is the (maximum) number of units of date 0 income that agent $i$ is prepared to give up to obtain 1 more unit of income in state $s$: we call this the *present value* of agent $i$ for income in state $s$ when the agent's consumption is $x^i$. The first-order conditions above can also be written as

$$q_j = \sum_{s=1}^S \pi_s^i(x^i) V_s^j, \qquad j = 1, \ldots, J \tag{8}$$

or in more condensed form as $q = \pi^i V$. Only when markets are complete (rank V=S) will agents agree on the present value of income in each state, and hence on the present value of all date 1 income streams. When markets are incomplete (rank $V < S$) there are many vectors $\pi = (\pi_1, \ldots, \pi_S)$ which satisfy $q = \pi V$ so that agents will typically not agree on the present value of income streams lying outside the span of the markets, $y \notin \langle V \rangle$.

In Drèze's paper the securities consist of the equity contracts of the $J$ firms, so that $V^j = y_{\mathbf{1}}^j$ where $y_{\mathbf{1}}^j$ is the date 1 profit stream of the $j^{\text{th}}$ firm, for $j = 1, \ldots, J$. Studying the allocations $(x, z, y) = (x^1, \ldots, x^I, z^1, \ldots, z^I, y^1, \ldots, y^J)$ consisting of consumption, portfolio and production plans which are constrained feasible, he finds that the criterion which comes out of the FOCs for a constrained optimum is that firm $j$ should choose the feasible production plan $y^j$ which maximizes the average present value of its production plan for its shareholders

$$\sum_{i=1}^I z_j^i \, \pi^i(\bar{x}^i) \, y_{\mathbf{1}}^j - y_0^j \tag{9}$$

each shareholder being weighted by his share of ownership $z^i_j$. The choice of a production plan for each firm is akin to the choice of a public good for its shareholders, for each shareholder can have a different valuation: the criterion (9) is the Lindahl criterion for the choice of the public good, noting that agent $i$ only "consumes" the share $z^i_j$ of the public good instead of consuming it all as in the standard public good problem. Using the average present-value vector of its shareholders permits the firm to value any production plan $y^j$ in its production set, including those for which $y^j_{\mathbf{1}} \notin \langle \bar{V} \rangle$, where $\bar{V}$ is the payoff of the firms' equities at equilibrium. Thus Drèze extended the notion of a stock market equilibrium to economies in which firms have general convex production sets, requiring that the production plan of each firm maximize the weighted present value (9). Since this criterion is derived from the FOCs for a constrained Pareto optimum, the equilibrium satisfies the first-order conditions for constrained optimality. However these FOCs are not sufficient to ensure constrained optimality: for the date 1 constrained feasible consumption streams are defined by $x^i_s = \omega^i_s + \sum_j y^j_s z^i_j$, where both $y^j_s$ and $z^i_j$ are choice variables, and thus the constrained feasible set is not convex. Drèze gave interesting examples of stock market equilibria which are not constrained optimal (see also Dierker-Dierker-Grodal (2002)).

The corporate form was originally introduced to create an institution which is permanent for society and liquid for individuals. As a result a corporation—which in principle lives forever—is owned by a sequence of temporary owners. In a setting where markets cannot value all the alternative production plans, Drèze's normative analysis suggests that the firm should choose its production plan in the interest of the shareholders who will receive its profit—in the two-period model the agents who are the new shareholders, after trading shares on the stock market. The firm's manager would thus need to correctly anticipate

(a) who these shareholders will be (the $z^i_j$'s)

(b) the present-value vectors $\pi^i$ of these agents.

Finding (a) and (b) is in a sense more difficult than solving a standard public good problem, for the manager not only needs to know the preferences of his constituents (the $\pi^i$'s) but he also needs to know who these constituents are (the $z^i_j$'s). In a multi-period economy in which shareholders change at each date, the problem only gets worse: anticipating the future shareholders and their preferences at each date-event would require a level of foresight beyond even the most omniscient manager. Thus, while Drèze's approach has strong normative appeal, the information required to implement such an equilibrium seems excessive. This led Grossman and Hart (1979, Chapter 12) to modify the approach proposed by Drèze, arguing that the best that a manager can be expected to do is to take into account the interests of the current shareholders who own the firm at the time the investment decision is made.

In the two-period model agents have initial ownership shares—$\delta_j^i$ denoting agent $i$'s initial ownership share in firm $j$—and trade at the initial date to obtain their "new" portfolios $z_j^i$, $i = 1, \ldots, I$, $j = 1, \ldots, J$ . The justification of the Drèze criterion hinges on the fact that agent $i$ will receive a share $z_j^i$ of firm $j$'s profit at date 1. Grossman-Hart (GH for short) propose instead that the manager of firm $j$ choose a production plan which maximizes the average present value of the plan for the initial shareholders

$$\sum_{i=1}^{I} \delta_j^i \, \pi^i(\bar{x}^i) \, y_{\mathbf{1}}^j - y_0^j \tag{10}$$

each shareholder being weighted by his initial ownership share $\delta_j^i$. Since this criterion does not lead to constrained Pareto optimality, GH need to show that maximizing (10) is in fact in the best interest of the original shareholders, informational constraints preventing the manager from taking into account the preferences of future shareholders of the corporation.

Since an initial shareholder may sell all or part of his equity before the production plan yields its profit, GH must explain how such a shareholder anticipates that the market would value his share, if the firm were to choose an alternative production plan outside the span of the markets at equilibrium. They introduce "out-of-equilibrium" expectation functions, which they call "competitive price perceptions", according to which agent $i$'s expectation of the market value $q_j^i$ of the firm, if it chooses a production plan $y^j$, is $q_j^i = \pi^i(\bar{x}^i) y_{\mathbf{1}}^j - y_0^j$. Thus the shareholder draws on his own valuation $\pi^i$ to fill in the missing valuation of the market. A social welfare argument similar to that of Drèze, but restricted to the initial shareholders, leads to the present-value criterion (10).

The informational requirements of the GH approach are still high: though (a)— namely who the shareholders are—which needs to be anticipated in the Drèze approach is known in the GH approach, their preferences (b) still have to be elicited: basically a public good problem needs to be solved by each firm. While GH argue that the criterion can be used in a multiperiod setting, this assertion rests on the tenuous assumption that the production plan chosen by the current shareholders cannot be amended later by future shareholders: without this assumption there is a problem of time consistency of the original plan.

The papers of Drèze and Grossman-Hart illustrate strikingly the difficulties posed by the problem of decision making in a production economy when prices can not play their "informational role" due to the incompleteness of the markets. Information about the value of "commodities", which is normally transmitted by prices, needs to be elicited from the "consumers" (of the profit streams) and this is likely to create incentive problems and to be costly—very costly when the consumers consist of an ocean of endlessly changing shareholders.

## IV. Sub-optimality of Competitive Equilibrium

After Diamond's paper it was accepted in the GEI literature that the appropriate criterion for evaluating the efficiency of a system of competitive markets must incorporate into the constraints the limited ability of the financial markets to redistribute income. Thus markets must be evaluated by the criterion of constrained Pareto optimality (CPO) rather than by the traditional (first best) Pareto optimality. The papers of Diamond and Drèze studying CPO allocations in a production economy were placed in the setting of a one-good, two-period model. In an exchange economy, or in a production economy in which partial spanning holds, a GEI equilibrium of a one-good, two-period economy is CPO. The message of the papers in this part is that, as soon as there are two or more goods, if the financial markets are incomplete then typically GEI equilibria cease to be constrained Pareto optimal.

In an exchange economy the idea of constrained optimality is to see whether a planner, by choosing the agents' portfolios instead of letting the agents choose them in competitive financial markets, could improve on the equilibrium allocation. In a one-good economy, once the planner has chosen the portfolios $(z^1, \ldots, z^I)$ of the agents, the consumption in each state is determined $(x_s^i = \omega_s^i + V_s z^i)$: it is easy to check that in this case the planner cannot find feasible portfolios $(\sum_i z^i = 0)$ which improve on an equilibrium allocation for all agents. However when there are two or more goods, the choice of portfolios no longer determines the date 1 consumption streams. If the planner were to choose directly the date 1 consumption of goods, then the constraint on income distribution through portfolios of the existing securities would be lost. For this reason, Stiglitz (1982, Chapter 13) who introduced the concept of constrained optimality which has subsequently been adopted in the literature for the multigood case, assumed that the date 1 consumption streams are the result of trade on the spot markets, agent $i$ 's income in state $s$ being $p_s\omega^i_s + V_s(p)z^i$ (endowment income plus portfolio income), where the portfolio $z^i$ has been chosen by the planner. In a production economy the "constrained" planner may in addition choose the firms' production plans, and depending on the paper, there are more or less severe restrictions on the date 0 transfers that the planner can use. If using the feasible instruments, it is not possible for a constrained planner to improve on the equilibrium, then the equilibrium allocation is said to be *constrained Pareto optimal* (CPO).

Stiglitz (1982, Chapter 13) studies a simple Diamond-type economy with an input at date 0, two goods at date 1, each produced by a separate firm with constant returns. One firm is risky, the other is riskless: thus agents can vary their risk and return by appropriately investing in the two firms. Stiglitz compares the FOCs satisfied by a GEI equilibrium with the FOCs for maximizing a weighted sum of the agents' indirect utility functions—which depend on the spot prices at date 1 and the income derived from portfolios—under the feasibility condition on agents' portfolios. He carefully enumerates all special cases where the FOCs coincide, so that a GEI is CPO, and concludes that these cases are "exceptional". In an important contribution (1986, Chapter 14) Geanakoplos and Polemarchakis (GP for short) showed how much greater conceptual clarity could be brought to bear on the problem of establishing inefficiency of GEI equilibria by introducing the techniques of differential topology. Focusing on the case of exchange economies with a fixed incomplete numeraire[8] asset structure, they showed that in a family of economies parametrized by agents' preferences and endowments, a GEI equilibrium of a "generic" economy, obtained by drawing the parameters at random, is not CPO.[9] The analysis is extended to general production economies in Geanakoplos-Magill-Quinzii-Drèze (1990, Chapter 15), GMQD for short.

The reason why a GEI equilibrium with many goods is not CPO can be explained as follows. By changing either the agents' portfolios, or the firms' investments, or both, the planner can change the income distribution and/or the supply of goods at date 1. Under appropriate conditions—for example, in an exchange economy agents must have different preferences—the change in the income

[8] A real security structure for which each security pays off amounts $A^j_s \in \mathbb{R}$ of the same good—the "numeraire" good—is called a *numeraire* asset structure.

[9] Several assumptions, including an upper bound on the number of agents, is needed to obtain the result: see the paper for a precise statement of the theorem.

distribution and/or the supplies of the goods changes the spot prices in the different states at date 1. When the equilibrium is not Pareto optimal, i.e when the present-value vectors of the agents are not equalized, there are changes in spot prices which can increase the welfare of all agents. The formal analysis of the way changes in agents' portfolios or in firms' investments lead to a change in social welfare is given in GP or GMQD's papers, or the survey paper by Magill-Shafer (1991). The reader can follow the full details in these papers.

Here we present a simple example which illustrates the main ideas in a transparent way. The example is a two-period version of a growth model studied in the macroeconomic literature (Aiyagari (1994)): we will show how changing the agents' portfolios (here their savings) can induce a change in spot prices (the wage rate and the price of capital) which can lead to an increase in social welfare from the equilibrium allocation when the present-value vectors of the agents are not equalized—namely when markets are incomplete.

Consider a two-period economy ($t = 0, 1$) with $I$ agents. At date 0 there is only one good which can either be consumed or transformed into capital, usable at date 1. At date 1, a firm using this capital and labor produces a consumption good, with the production function $y = F(K, L)$, exhibiting constant returns to scale and satisfying the Inada conditions $F_K(K, L) \to \infty$ if $K \to 0^+$ and $F_L(K, L) \to \infty$ if $L \to 0^+$. Agents have preferences only for the consumption good, and have a standard utility function

$$U(x_0, x_1) = u(x_0) + \beta E(u(x_1))$$

where $0 < \beta \leq 1$, $u$ is strictly concave increasing and $x_1$ is the (possibly) random consumption at date 1. Each agent is endowed with $\omega_0$ units of the good at date 0 and with labor at date 1.

Suppose first that there is no uncertainty in the agents' labor endowments at date 1: each agent has the labor endowment $\bar{\ell}$. The equilibrium is characterized by the typical agent's choice of capital $k$ to carry over to date 1, the wage $w$ and the price $R$ of capital at date 1, satisfying the following conditions

$$x_0 = \omega_0 - k, \quad x_1 = w\bar{\ell} + Rk, \quad u'(x_0) = \beta u'(x_1)R \tag{11}$$

$$w = F_L(K, L), \quad R = F_K(K, L), \quad K = kI, \quad L = \bar{\ell}I \tag{12}$$

(11) is the FOC for the optimal choice of $k$, while (12) gives the FOC for profit maximization and the market clearing conditions. Since agents can use capital (savings) to redistribute income over

time, the financial markets are complete.

Suppose that a planner changes the capital chosen by the representative agent by $dk$, and lets the markets at date 1 adapt to this change. Then

$$dx_0 = -dk, \quad dx_1 = dw\bar{\ell} + dRk + Rdk$$

$$dw = F_{LK}(K,L)Idk, \quad dR = F_{KK}(K,L)Idk \tag{13}$$

Since $F_K$ is homogenous of degree 0, $dw\,\bar{\ell} + dR\,k = 0$, so that $du = u'(x_0)(-dk) + \beta u'(x_1)R\,dk$, which is zero in view of the FOC (11). Thus when markets are complete the planner cannot improve on the equilibrium outcome by changing the choice of investment at date 0.

Now suppose that each agent faces uncertainty about his possible labor endowment at date 1. At the beginning of date 1 nature draws $n$ agents who are given $\ell_b$ units of (effective) labor, the remaining $I-n$ agents being given $\ell_g$ units of labor, with $\ell_b < \ell_g$. There are thus $\begin{pmatrix} n \\ I \end{pmatrix} = \dfrac{I!}{n!(I-n)!}$ aggregate states of nature, all equiprobable, which differ from one another by the names of the agents who have the good and the bad draw of their labor endowment. In every state the total supply of labor is the same, $L = n\ell_b + (I-n)\ell_g$. From the point of view of an agent all the states in which he/she has a good draw are equivalent, so that each agent perceives a probability $\rho = \dfrac{n}{I}$ of having a bad draw and $1-\rho$ of having a good draw. There are no insurance markets against these labor risks. The equilibrium $(k, w, R)$ is characterized by

$$x_0 = \omega_0 - k, \quad x_b = w\ell_b + Rk, \quad x_g = w\ell_g + Rk \quad u'(x_0) = \beta(\rho u'(x_b) + (1-\rho)u'(x_g))R \tag{14}$$

$$w = F_L(K,L), \quad R = F_K(K,L), \quad K = kI, \quad L = (\rho\ell_b + (1-\rho)\ell_g)I \tag{15}$$

If the planner changes the investment at date 0 by $dk$ then

$$dx_0 = -dk, \quad dx_b = dw\ell_b + dRk + Rdk, \quad dx_g = dw\ell_g + dRk + Rdk$$

while the change in prices $(dw, dR)$ is given by (13). The induced change in utility is

$$du = u'(x_0)(-dk) + \beta(\rho u'(x_b)dx_b + (1-\rho)u'(x_g)dx_g)$$

Substituting $(dx_0, dx_b, dx_g)$ the direct effect of the change $dk$ is zero in view of the first-order condition in (14) (since $k$ is chosen optimally at equilibrium) but the price effects remain

$$du = \beta\left[(\rho u'(x_b)\,\ell_b + (1-\rho)u'(x_g)\,\ell_g)\,dw + (\rho u'(x_b) + (1-\rho)u'(x_g)\,)k\,dR\right]$$

Let $\bar{\ell} = \rho\ell_b + (1-\rho)\ell_g$ denote the mean labor endowment. Since $dw\,\bar{\ell} + dR\,k = 0$, the terms in $dw$ and $dR$ would cancel if $u'(x_b) = u'(x_g)$ i.e. in the case of certainty or the case with full insurance. In the absence of insurance markets, $u'(x_b) \neq u'(x_g)$ and $du \neq 0$. Letting $\boldsymbol{\ell_1}$ and $\boldsymbol{x_1}$ denote the random labor endowment and random consumption, $du$ can be written as

$$\begin{aligned} du &= \beta(E(u'(\boldsymbol{x_1})\boldsymbol{\ell_1})dw + E(u'(\boldsymbol{x_1})kdR) \\ &= \beta(E(u'(\boldsymbol{x_1})E(\boldsymbol{\ell_1})dw + +cov(u'(\boldsymbol{x_1}, \boldsymbol{\ell_1})dw + E(u'(\boldsymbol{x_1}))kdR) \\ &= \beta\,cov(u'(\boldsymbol{x_1}), \boldsymbol{\ell_1})dw \end{aligned}$$

Since $u'$ is decreasing, it follows that $cov(u'(\boldsymbol{x_1}), \boldsymbol{\ell_1}) < 0$. A change $dk < 0$, which implies $dw < 0$, leads to an increase in welfare: $du > 0$.

Reducing saving at date 0 increases date 0 consumption and reduces consumption at date 1, and to terms of first order, the direct effect of the change in consumption is zero, since agents have optimized on their choice of saving at equilibrium. But the price of capital increases and the price of labor decreases, shifting the representative agent's income away from the risky labor income $(w\ell_g, w\ell_g)$ and towards the sure return $(kR, kR)$ on capital. The price effect reduces the variability of date 1 consumption, improving the welfare of the representative agent. The change in prices (partially) replaces the insurance market which is missing.

The presence of systematic inefficiencies in the choice of portfolios by consumers and investment by firms when markets are incomplete suggests the possibility that intervention by a well-informed "planner" could improve on the functioning of the markets. In GP and GMQD the "constrained planner" was used to show that changing the choices of the maximizing agents facing competitive markets can improve social welfare, but it was not meant to suggest that this type of intervention is feasible. Altering agents' portfolios is almost tantamount to closing the financial markets and letting the planner choose the income transfers among agents, but this would require so much detailed information that the overall cost would be prohibitive. Thus the literature on constrained efficiency has progressively shifted away from the proof of constrained sub-optimality of equilibrium towards a search for realistic policy instruments which can improve on the equilibrium.

The approach used to study constrained sub-optimality of equilibrium in GP and GMQD is based on a reduced-form approach to the equilibrium equations which takes as given the agents' demand functions. A more flexible and powerful approach is proposed by Citanna-Kajii-Villanacci

(1998, Chapter 16) who include the first-order conditions and budget equations implicitly defining the agents' demands in the equations of equilibrium. They use their approach to show that well chosen lump-sum taxes and transfers among the agents (rather than changes of portfolios) can be used to improve on an equilibrium outcome. The recent paper of Citanna-Polemarchakis-Tirelli (2006) uses the same approach to show that improvements can be obtained by imposing taxes and subsidies on the purchase of securities—this is indeed a natural way of inducing agents to change their portfolio holdings and hence of influencing the distribution of income and the spot prices at date 1. In the simple example described above, taxing the return to capital and redistributing the proceeds in a lump sum way would improve on the equilibrium. However the result that taxes on securities can be used in general exchange economies to improve the welfare of all agents requires an assumption on the relative number of securities and agents (less agents than securities), since there must be a sufficient number of "instruments" (taxes on securities) relative to the number of "objectives" (agents' utilities).

## V. Nominal Securities, Indeterminacy, Sunspots, and Real Effects of Money

In the GEI model agents purchase financial securities to redistribute their income across dates and states, and in each date-event they spend the income to purchase goods on the spot markets. We mentioned earlier that it is useful to distinguish two basic types of securities: *real* securities which promise to deliver the spot-market value of a bundle of goods, and *nominal* securities which promise to deliver specified amounts of a "unit of account". It should not come as a surprise that the equilibria of the model with real assets behave differently from the equilibria of the model with nominal securities: in the model with real assets the equilibrium allocations are typically determinate (finite in number), while in the model with nominal assets they may be indeterminate (there is a continuum of possible equilibrium allocations). Simple economic intuition suggests the reason. A real security is a contract which promises a payoff which is proportional to the spot prices in each state: doubling the spot prices will double its payoff. Real securities are inflation proof, so that only the relative prices of goods in the spot markets matter for the equilibrium. As a result, like in the standard general equilibrium model, the equilibria of a GEI model with real securities are determinate: for given characteristics of the economy, the model predicts a finite number of possible equilibrium allocations (there is however nominal indeterminacy since the price levels are not determined).

On the other hand if securities are nominal, doubling the price level in one state halves the purchasing power of the income promised by the assets in this state. If the markets are incomplete and the structure of securities is not sufficient to permit income to be transferred exclusively to this state, then agents may not be able to "undo" the change in expected inflation in this state. The same economy, with the same nominal payoffs for the securities, can have different equilibrium outcomes depending on the price levels on the spot markets at date 1.

This indeterminacy of the equilibrium outcomes of an economy with nominal securities was first discovered by Cass (1989) who introduced the GEI model with nominal securities (Cass 2006, Chapter 6). The results that have been obtained for economies with nominal assets are among the most interesting and controversial in the theory of incomplete markets. Basically there are two camps: those who, like Cass, advocate the idea that nominal assets capture the inherent indeterminacy of GEI equilibrium because of the dependence of the equilibrium on beliefs, modeled by the presence of "sunspots" (the sunspot camp); on the other side those who, like us, think that nominal assets make sense only in a monetary economy in which money is explicitly introduced to tie down the price level (the money camp). As with all good controversies, depending on where one wants to start the analysis, each camp has a strong argument supporting its point of view.

The common point of departure can be found in the papers of Balasko-Cass (1989) and Geanakoplos-Mas-Colell (1989, Chapter 17) which provide a systematic study of the degree of indeterminacy of the real equilibrium allocations of an economy with nominal assets and incomplete markets, when the equilibrium is defined in the usual way by equality of supply and demand on the goods and security markets. They show that there is a striking difference between the structure of the equilibria depending whether the markets are complete or incomplete. In both cases there is nominal indeterminacy—this simply reflects the fact that in the standard definition of a GEI there is no condition introduced to tie down price levels. When markets are complete however equilibrium allocations are determinate, typically finite in number and locally unique. But when markets are incomplete equilibrium allocations are indeterminate and, depending on how the indeterminacy is parametrized, it translates into different dimensions for the real equilibrium outcomes.

The simplest result to understand is that of Geanakoplos-Mas-Colell (GMC for short) who find that under appropriate conditions the dimension of indeterminacy is $S-1$. They parametrize the equilibria by the spot prices of one of the goods, say good 1, in each state. While there are $S+1$ parameters, two can be fixed without affecting the equilibrium allocation, so that at most $S-1$ parameters remain. For, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium, if all date 0 prices are multiplied by 2, then the budget set (1) of each agent is unchanged so that $(\bar{x}, \bar{z}, (2\bar{p}_0, \bar{p}_{\mathbf{1}}), 2\bar{q})$ is also an equilibrium with the same allocation $\bar{x}$. Thus prices at date 0 can be normalized by setting $p_{01} = 1$. On the other hand if all prices in all states at date 1 are doubled, the purchasing power of all securities is halved. If, as a result, the price of each security is halved, each agent can buy twice the original amount of the securities and obtain the same real income transfer. Thus $(\bar{x}, 2\bar{z}, (\bar{p}_0, 2\bar{p}_{\mathbf{1}}), \bar{q}/2)$ is an equilibrium with the same allocation $\bar{x}$. The spot prices at date 1 can be normalized by choosing (for example) $\sum_{s=1}^{S} p_{s1} = 1$. In economic terms expected inflation can be incorporated into the prices of the nominal securities and has no effect on the equilibrium allocation, but the variability of inflation may have real effects.

GMC show that the equilibria of the economy with nominal securities are the union of the equilibria of economies with real securities where the $j^{\text{th}}$ security delivers $V_s^j/p_{s1}$ units of good 1, for $j = 1, \ldots, J,\ s = 1, \ldots, S$, for all possible parameters $(p_{s1}) \in \mathbb{R}^S$, with $\sum_{s=1}^{S} p_{s1} = 1$. For a given value of the parameters $(p_{s1})$, let $V_p$ denote the associated payoff matrix of the real securities. GMC show that, if $J < S$ and if the original matrix $V$ of the nominal securities is in general position, i.e. each $J \times J$ submatrix has rank $J$, then two different vectors of parameters $(p_{s1})_{s=1}^{S} \neq (p'_{s1})_{s=1}^{S}$ imply $\langle V_p \rangle \neq \langle V_{p'} \rangle$. If agents use the full subspace of income transfers, which is a generic property at equilibrium, then altering the relative prices of good 1 changes the possible income transfers and hence the equilibrium allocation.

Balasko-Cass (1989) use a different parametrization, fixing exogenously the prices of the securities and then parametrizing by the present-value vector of agent 1, which can vary in a subspace of dimension $S-J$ since it must satisfy the no-arbitrage equations (2). Under an assumption guaranteeing that the matrix of security payoffs does not have too many zeros, they show that generically in agents' endowments, the equilibrium allocations have dimension $S-J$. If the prices of the securities are also included as parameters, then, since their sum can be normalized, the degree of indeterminacy comes back to being $S-1$ as in GMC.

Originally Cass was interested in economies with incomplete financial markets and nominal securities as a way of generating sunspot equilibria. These are essentially stochastic equilibria of a non-stochastic economy which exploit the incomplete co-ordination between the financial markets for transferring income and spot markets for allocating goods. Sunspot equilibria are equilibria in which the consumption of the agents are different in two apparently fictitious states, in which there is no difference in either agents' preferences or the endowments of goods. Cass (1992, Chapter 18) shows that economies with nominal assets in which financial markets are incomplete (there is not perfect insurance against "sunspots") generate such equilibria. For example if there is a nominal bond which pays 1 unit of account at date 1, if agents anticipate different price levels for $S$ "sunspot" states, then the economy can generate equilibrium allocations of dimension $S-1$. If there are additional securities with payoffs dependent on sunspots, then generically in the agents' endowments, the dimension of these equilibria is $S-J$.

Gottardi-Kajii (1999, Chapter 19) exploit the framework of Cass (1992, Chapter 18) to take the analysis of sunspot equilibria one step further. They show that an economy with deterministic preferences and endowments, and with real securities having deterministic commodity payoffs, can generate sunspot equilibria. The idea is simple: they associate with any two-period deterministic economy, an "artificial economy" with two sunspot states and a single security which pays a different amount $(V_1, V_2)$ in the two states. The analysis of Cass shows that if agents have different preferences for the goods then differences in the income distribution in the two states implies different spots prices ($p_1 \neq p_2$) and allocations ($x_1 \neq x_2$) in the two states. If there are two or more goods there exists a bundle $A \in \mathbb{R}^L$ of the goods such that $p_1 A = V_1$ and $p_2 A = V_2$. The economy with one asset with real payoff $A$ and two sunspot states has a sunspot equilibrium, even though the fundamentals, preferences, endowments and asset payoffs, are deterministic.

Sunspot equilibria attempt to formalize an intuition that economists have long held, but have found difficult to formalize, namely that "beliefs" can in and of themselves influence the equilibrium outcome. The idea that "beliefs" can influence the equilibrium outcome and be "self-fulfilling" is a very appealing idea. However when sunspot equilibria are derived from a model with nominal assets, they exploit an indeterminacy in the standard concept of equilibrium, namely the indeterminacy of price levels, and sunspots serve to co-ordinate the anticipations of the agents on price levels which nothing else determines. Given that we have yet to find a convincing example of "sunspots", it seems hard to believe that agents trading nominal securities will use sunspots to co-ordinate their anticipations of future price levels.

Our paper (Magill-Quinzii, 1992, Chapter 20) takes a different perspective on the appropriate way of exploring the model with nominal securities. Our first observation is that in the real world a nominal contract is a promise to make a deferred payment of a sum of money at a future date: such promises only come to be made in an economy in which money is already used as medium of exchange and as a unit of account. The first step is therefore to find a natural way of introducing money as a medium of exchange into the GEI model so that price levels are determined by the monetary side of the economy.

The basic idea of a monetary model in the style of Clower (1967) can be described as follows. In a barter economy an agent wanting to exchange good $\ell$ for good $\ell'$ must find another agent who is simultaneously willing to perform the opposite transaction. Introducing money as a medium of exchange avoids the problem of double coincidence of wants by splitting the exchange into two separate transactions: good $\ell$ is sold for money, and then money buys good $\ell'$. Thus we model the use of money in facilitating transactions by dividing each period into sub-periods: agents first sell their goods (initial endowments) in exchange for money, then they transact on the security markets, increasing their money balances if they are net borrowers, or decreasing them if they are net lenders. Finally they use the money acquired from these transactions to buy their desired consumption bundles. At date 1 in each state the same process takes place, agents getting the payoffs from their portfolios, instead of paying for them. To ensure that money has positive value in a finite horizon economy we introduce a Central Exchange which pays agents money for their goods at the beginning of each period and recovers the money when agents buy their goods at the end of the period, $M_s$ being the money injected in date-state $s$, $s = 0, \ldots, S$. Equating the agents' transaction demand for money in each date-state with the money supplied by the monetary authority leads to the $S + 1$ quantity theory equations

$$\sum_{i=1}^{I} p_s x_s^i = M_s, \qquad s = 0, \ldots, S$$

This timing of transactions using the Central Exchange to inject money before agents buy their goods is different from the standard timing of the cash-in-advance model in which agents first buy goods for money and later receive payment for their endowments. However it has the advantage of leading to the same budget equations as in a GEI equilibrium, while introducing precisely the requisite number of equations to determine the $S+1$ price levels.

In Magill-Quinzii (1992, Chapter 20) we take the characteristics of an economy to be the agents' utility functions $u = (u^1, \ldots, u^I)$ and endowments $\omega = (\omega^1, \ldots, \omega^I)$, the matrix of payoffs $V$ of the nominal securities, and the monetary policy $M = (M_0, M_1, \ldots, M_S)$. In the spirit of the rational expectations literature agents are assumed to correctly anticipate the monetary policy and hence to correctly anticipate the price levels in the different states at date 1. As a result a given monetary economy $(u, \omega, V, M)$ typically has a finite number of equilibria. We say that monetary policy is *non-neutral* (or has *real effects*) if a different monetary policy $M'$ can lead to an equilibrium with different consumption streams for the agents. We show that if markets are complete monetary policy is neutral while, if markets are incomplete, then (generically in endowments) monetary policy has real effects. Thus the indeterminacy of equilibrium in the GEI model without price level determination, becomes the property that, in an economy with nominal assets and incomplete markets, correctly anticipated monetary policy has real effects.

Another approach to modeling money in a GEI model is presented in the paper of Dubey-Geanakoplos (2003, Chapter 21) which adopts the more classical timing of the Lucas (1980) cash-in-advance model in which agents must have money to buy their goods before they receive the monetary payment from the sale of their endowments. A "bank" lends money to agents at date 0 against bonds which mature either at the end of the first or the second period, and again in each state $s$ at date 1 against bonds which mature at the end of this period. All the money in the system must return to the bank at the end of the second period. If agents have initial money balances, more money must come back to the bank than it lends, so that the interest rates charged by the bank must be positive. This ensures that money has positive value and that equilibria with either nominal or real securities are determinate. However with a positive interest rate the model is not exactly the GEI model: the buying prices of goods exceed their selling prices since money must be borrowed at a positive interest rate to buy goods. GEI equilibria obtain in the limit when the quantity of money injected by the bank goes to infinity. The emphasis of the paper is less on determinacy and real effects of money that on the fact that an equilibrium exists even when the securities are real and the payoff matrix has potential changes of rank—the problem which, as we have seen above, can lead to non-existence of equilibrium in the standard GEI model. Explicitly introducing money into the model and requiring that agents have the money to buy securities before they receive the payment for the securities they sell, limits agents' transactions on the security markets: essentially they create the bounds that Radner postulated, leading to the existence of an equilibrium.

Drèze and Polemarchakis (1999, 2001) have developed a version of the above model in which the bank is private and makes a profit from lending money for transactions. In later papers they reinterpret this profit as seignorage revenue earned by the government and are led to enter the debate on the fiscal theory of the price level (Bloise-Drèze-Polemarchakis (2005)). These are certainly interesting issues which bring the model close to the macroeconomic literature, but they are not directly related to the incomplete markets agenda on which we focus here.

## VI. Security Structure

The GEI model provides a systematic framework for studying the consequences of having only limited securities for coping with future contingencies. An inherent weakness of the model however is its limited ability to justify an existing security structure. Considerable interest has therefore centered on finding ways of analyzing the benefits (if any) that can be expected from adding new assets to an existing security structure. Changing the number of assets from $J$ to $J+1$ typically changes the subspace of income transfers to which agents have access and—if this is done in a "brutal" way—will lead to a discontinuous change in the equilibrium, rendering 'before' and 'after' comparisons difficult. Elul (1995) and Cass-Citanna (1998, Chapter 22) discovered (essentially simultaneously) a clever way of embedding a new security into an existing equilibrium in such a way that there is a smooth transition between the original equilibrium and a neighboring equilibrium with $J+1$ assets and an augmented span.

The basic idea is to begin by introducing a new asset which does not change the equilibrium; the payoff of the asset is then perturbed in such a way that the equilibrium changes smoothly, permitting techniques of differential topology to be used to compare neighboring equilibria. The type of asset which does not change the equilibrium is either a redundant asset as in Cass-Citanna (1998, Chapter 22) or an asset that no agent wants to trade at the original equilibrium as in Elul (1995, (1999, Chapter 23)).

The technique of Cass-Citanna in Chapter 22 is to write the system of equations defining an equilibrium with $J+1$ securities. When the $J+1$'st security is redundant the original equilibrium is a solution of this system of equations. They then study the Jacobian of the system at this solution and show that if the present-value vectors of the agents are different—a generic property of an incomplete market equilibrium—then the Jacobian is of maximum rank. It follows that there are changes in the payoff of the new security for which the equilibrium changes smoothly. Finally they show that if the number of (types of) agents is small relative to the degree of incompleteness of the market, then generically in preferences and endowments, any local change $\Delta u = (\Delta u^1, \ldots, \Delta u^I)$ in agents' utilities can be achieved by a suitable perturbation of the new asset. Thus depending on the particular payoff of the new asset, its introduction can benefit all agents, harm all agents, or improve some and harm others.

The ambiguity in the consequences of introducing a new asset has its origin in the numerous ways in which prices can change following the introduction of the asset. More precisely, adding a new asset to an existing security structure has two effects on the equilibrium:

(i) the *direct effect*, namely the increase in the span of the market; this always increases the welfare of the agents

(ii) the *indirect*, or *price effect*: changing the agents' income transfers leads to a change in the distribution of income among the agents trading on the spot and security markets and, when agents have different preferences, this induces a change in the spot and security prices. These price changes can either increase or decrease agents' utilities.

The papers of Elul (1995) and Cass-Citanna (1998, Chapter 22) mentioned above consider the general case where both effects are present, and the price changes consist of changes in both the spot and the security prices. By restricting the possible price changes, more specific results can be obtained: for example Elul (1999, Chapter 23) studies the two-period, one-good (or finance) model, which eliminates the spot-price changes. The only remaining price effect which can counteract the direct effect is the change in the prices of other securities which arise when a new security is introduced. Elul shows that provided that there are not too many agents relative to the degree of incompleteness of the markets, it is possible to introduce a new security which does not change the prices of the existing securities, so that only the direct effect is present and the welfare of all agents is increased.

Instead of asking whether adding a security to an existing security structure would improve the welfare of all agents, one could more generally ask whether there is an optimal way of adapting the entire security structure $V$ to the risks $\omega = (\omega^1, \dots, \omega^I)$ faced by the agents, given the risk tolerances summarized by their utility functions. This is the question studied by Demange-Laroque (DL) (1995, Chapter 24) in the specific setting of a one-good model with normally distributed endowments. Implicit in their analysis is the fact that in a two-period one-good exchange economy with security structure $V$, a financial market equilibrium is $V$-constrained Pareto optimal. Furthermore with income transfers among the agents, any $V$-constrained Pareto optimal allocation can be realized as a financial market equilibrium. Thus DL focus on the constrained optimal allocations and study the following problem: consider the family of all incomplete security structures with $J$ assets; among the induced family of $V$-constrained Pareto optimal allocations, find the one which generates the greatest social welfare.

Answering this question in a framework of any generality is difficult, so DL focus on a setting where the problem can be resolved in a clear and simple way: agents have constant-absolute-risk-aversion (CARA) utility functions and their endowments are normally distributed. More precisely they consider a model in which agents only consume at date 1 and have random endowments of the form

$$\omega^i = \mathbf{1}\bar{\omega}^i + \widetilde{\phi} w^i$$

where $\mathbf{1}$ is the constant random variable equal to 1, $\bar{\omega}^i$ is the agent's mean endowment, and $\widetilde{\phi} = (\widetilde{\phi}^1, \dots, \widetilde{\phi}^K)$ are $K$ independent normally distributed random variables (factors), each with zero mean and unit variance, which represent the (normalized) risks of the economy, $w^i = (w^i_1, \dots, w^i_K)$ representing agent $i$'s exposure to these risks. Each agent has a utility function of the form

$$U^i(x^i) = Eu_i(x^i) = E\left(-\frac{1}{a_i} e^{-a_i x_i}\right)$$

where $a_i = -\dfrac{u_i''(c)}{u_i'(c)}$ is the constant absolute risk aversion and $\tau_i = 1/a_i$ is the constant risk tolerance of agent $i$. The Pareto optimal allocations of this economy are of the form

$$x^{i*} = \mathbf{1}\bar{x}^i + \frac{\tau_i}{\tau}\widetilde{\phi} w, \quad i = 1, \dots, I \tag{16}$$

where $w = \sum_i w^i$ is the aggregate exposure to the risks, and $\tau = \sum_i \tau_i$ is the sum of the agents' risk tolerances. Thus each agent's consumption stream consists af a sure part $\mathbf{1}\bar{x}^i$ and a share of the aggregate risk $\widetilde{\phi} w$ equal to the agent's relative risk tolerance $\tau_i/\tau$. If all the risks of the economy are traded, i.e. if the security structure consists of the riskless bond with payoff $\mathbf{1}$ and of risky securities which span the subspace $\langle\widetilde{\phi}\rangle$, then agents can perfectly diversify their risks and, with appropriate (sure) income transfers, all first-best allocations can be obtained.

Suppose however that complete risk-sharing is not attainable, for reasons which are not explicitly modeled. Instead the risk-sharing possibilities are limited to any subspace $\langle\widetilde{V}\rangle$ generated by $J$ ($J < K$) random variables $(\widetilde{V}^1, \ldots, \widetilde{V}^J)$ (the payoff of the securities) which are linear combinations of the basic risks $\widetilde{\phi}$, so that $\langle\widetilde{V}\rangle \subset \langle\widetilde{\phi}\rangle$. Then it can be shown that, for a given security structure $(\mathbf{1}, \widetilde{V})$, the constrained Pareto optimal allocations are of the form

$$x^i = \mathbf{1}\bar{x}^i + (\widetilde{\phi} - E(\widetilde{\phi}|\widetilde{V}))w^i + \frac{\tau_i}{\tau}E(\widetilde{\phi}|\widetilde{V})w, \quad i = 1, \ldots, I \tag{17}$$

where $E(\widetilde{\phi}|\widetilde{V})$ denotes the conditional expectation of the random variables $(\widetilde{\phi}^1, \ldots, \widetilde{\phi}^K)$ given $(\widetilde{V}^1, \ldots, \widetilde{V}^J)$. This has a simple geometric interpretation. Consider the inner product on the linear space $\langle\widetilde{\phi}\rangle$ defined by $[x, y] = E(xy) = cov(x, y)$ (since the random variables in $\langle\widetilde{\phi}\rangle$ have zero mean). The conditional expectation $E(\widetilde{\phi}|\widetilde{V})$ is the orthogonal projection of the risks factors $\widetilde{\phi}$ on the subspace $\langle\widetilde{V}\rangle$ spanned by the securities, in the metric induced by this inner product. Thus the streams in (17) are the closest possible to the Pareto optimal allocations (16): the limited risk sharing possibilities force agents to keep the undiversifiable part $(\widetilde{\phi} - E(\widetilde{\phi}|\widetilde{V}))w^i$ of their endowment risks, while they share the aggregate diversifiable risks $E(\widetilde{\phi}|\widetilde{V})w$ in proportion to their relative risk tolerances.

DL then solve the following problem: given that consumptions streams have the form given in (17), find the subspace $\langle\widetilde{V}\rangle$ which maximizes social welfare. Because of the CARA form of the preferences and the normally distributed risks, they show that this is equivalent to maximizing

$$\sum_{i=1}^{I} a_i \left\| E(\widetilde{\phi}|\widetilde{V}) \left(\frac{\tau_i}{\tau}w - w_i\right)\right\|^2 \tag{18}$$

If the planner has no restrictions on the risk transfers that he can make among agents, he will achieve a first-best allocation by correcting the original exposure $\widetilde{\phi}w^i$ of agent $i$ to the factor risks by the net trade (in risks) $t^{i*} = \widetilde{\phi}\left(\frac{\tau_i}{\tau}w - w^i\right)$ so that agent $i$ has his appropriate share of the aggregate risks. If the planner is constrained to redistribute risks by using the random variables $\widetilde{V}$, then $t^{i*}_{\widetilde{V}} = E(\widetilde{\phi}|\widetilde{V})\left(\frac{\tau^i}{\tau}w - w^i\right)$ is the component of the optimal net trade $t^{i*}$ which can be achieved using the securities $\widetilde{V}$. The planner will choose $\langle\widetilde{V}\rangle$ so that the greatest possible share of the optimal risk correction is achieved. However, with a limited structure, choosing a subspace $\langle \widetilde{V} \rangle$ "close" to agent $i$'s optimal $t^{i*}$ may imply that the projection of the optimal trade $t^{i'*}$of some other agent $i'$ is small; the planner must thus weight each agent's needs, and (18) indicates that the appropriate weights are the agents' risk-aversion coefficients: the need to correct the risks of a risk-averse agent has more weight than that of a risk-tolerant agent. Thus the planner chooses the subspace $\langle \widetilde{V} \rangle$ so as the maximize the risk-aversion weighted sum of the projections of the optimal risk corrections

$$\sum_{i=1}^{I} a_i \left\| t^{i*}_{\widetilde{V}} \right\|^2$$

or, equivalently, so as to minimize the risk-aversion weighted sum of the unachievable risk transfers $\sum_{i=1}^{I} a_i \left\| (t^{i*} - t^{i*}_{\widetilde{V}}) \right\|^2$.

Calculating the projections leads to the conclusion that $\langle \widetilde{V} \rangle$ must be spanned by $J$ independent payoff streams of unit variance, whose components in the basis $\widetilde{\phi}$ are the $J$ eigenvectors of the symmetric matrix $\sum_{i=1}^{I} a_i (\frac{\tau_i}{\tau} w - w_i)^\top (\frac{\tau_i}{\tau} w - w_i)$ associated with the $J$ largest eigenvalues, each eigenvalue measuring the gain in social welfare from introducing trade in the direction of the corresponding eigenvector. Note that the criterion of choosing a security or a subspace so as to maximize the volume of trade—the criterion which would be chosen by a security exchange which derives its revenue from commissions on transactions (Duffie-Jackson (1989), Hara (1995))—goes in the right direction but, if there are differences in risk aversion among agents, it will not lead exactly to the optimal security structure since it maximizes the unweighted net trades. DL also show that the analysis of the CARA-normal case can be partially extended to economies with more general preferences as long as the risks are normally distributed. Results similar to those of DL in the CARA-normal case were derived independently by Athanasoulis-Shiller (2000).

The analysis of DL implicitly assumes that there is a bond with a riskless payoff and focuses on the choice of the risky securities. A suggestive way of looking at the Magill-Quinzii paper (1997, Chapter 25) is that it does the reverse: it takes the optimal structure of risky assets (equity) as given and asks, in the setting of a monetary economy, which bond can be introduced whose real payoff is as close as possible to riskless stream needed to obtain an efficient allocation of risk.

Although most finance models assume the existence of a (real) riskless bond, in practice most borrowing and lending activity occurs in nominal bonds whose real payoff fluctuates with the purchasing power of money (ppm). Indexation has long been advocated as the solution for isolating the real payoff of nominal bonds from the variability of the ppm, but the stylized fact is that the variability of inflation has to be substantial before agents switch systematically to indexed contracts. Our paper seeks to explain this phenomenon by arguing that indexation is never perfect and that a bond with a riskless payoff does not exist. For, if a bond is indexed on the value of a bundle of goods and there are aggregate risks which generate fluctuations in relative prices, such an indexed bond substitutes the fluctuations in relative prices of the goods in the reference bundle for the fluctuations in ppm affecting the payoff of a nominal bond. Depending on the relative magnitudes of these two different risks, a nominal or an indexed bond is optimal.

To formalize this argument we draw on the monetary multigood GEI model of Chapter 20, restricting the structure of preferences so that the spot prices can be explicitly calculated, and the interplay between the security structure and the welfare of agents in equilibrium can be studied in a simple and clear way. To this end agents are assumed to have separable-homothetic preferences of the form

$$U^i(x^i) = \lambda_0^i h(x_0^i) + \sum_{s=1}^{S} \rho_s u^i(h(x_s^i))$$

where $\lambda_0^i > 0$, $h : \mathbb{R}_+^L \to \mathbb{R}$ is concave, increasing, and homogenous of degree 1, $\rho_s$ denotes the probability of state $s$, and $u^i$ is quadratic, $u^i(c) = -\frac{1}{2}(\alpha^i - c)^2$. Thus all agents have the same homothetic preferences for the $L$ goods in each spot market, and have linear-quadratic preferences over the induced utilities $\left(h(x_0^i), (h(x_s^i))_{s=1}^S\right)$.

The existing risky securities consist of $K$ equity contracts with payoff matrix $V$, and these securities are well adapted to the risks of the agents in that agents' endowments lie in the span $\langle V \rangle$ of $V$. Given that a bond is needed to accommodate the differences in risk aversion of the agents, the problem is to find which bond to add—either a nominal or an indexed bond—to maximize social welfare at equilibrium.

In view of the assumption of identical homothetic preferences, the vector of spot prices in state $s$ is proportional to the marginal utilities of the representative agent at the aggregate endowment

$$p_s = \frac{M_s}{h(\omega_s)} \nabla h(\omega_s), \qquad s = 0, \ldots S$$

where $\omega_s = \sum_i \omega_s^i$ is the aggregate endowment in state $s$, $\nabla h = \left(\dfrac{\partial h}{\partial x_1}, \ldots, \dfrac{\partial h}{\partial x_L}\right)$, and $M_s$ is the quantity of money issued by the monetary authority in state $s$. If $m_s^i$ denotes the monetary income of agent $i$ in state $s$, the agent's consumption is $x_s^i = \frac{m_s^i}{M_s}\omega_s$ and hence the agent's indirect utility function over monetary income streams $(m_0^i, \ldots, m_S^i)$ is

$$\widetilde{v}^i(m^i) = \lambda_0^i\, h\left(\frac{m_0^i}{M_0}\omega_s\right) + \sum_{s=1}^{S} \rho_s u^i\left(h\left(\frac{m_s^i}{M_s}\omega_s\right)\right)$$

The coefficients

$$\nu_s = \frac{h(\omega_s)}{M_s}, \qquad s = 0, \ldots S$$

define the "real" or "utility" index of the purchasing power of money, so that $\mu_s^i = \nu_s m_s^i$ is agent $i$'s real income in state $s$. The homogeneity of $h$ leads to the indirect utility function for real income

$$v^i(\mu^i) = \lambda_0^i\, \mu_0^i + \sum_{s=1}^{S} \rho_s u^i(\mu_s^i)$$

Once converted into real purchasing power, the economy becomes a standard linear-quadratic finance economy for which the equilibrium social welfare can be evaluated. We show that, when equity contracts spanning the agents' endowments are traded, the equilibrium social welfare is proportional to the length of the projection of the riskless income stream onto the market subspace, with the same metric on random variables as in DL. If a risky bond is added to the equity contracts, the length of this projection depends on the variance of the risky bond and on its covariances with the equity contracts.

Suppose that the bond is nominal, paying one unit of money in each state. Then its payoff in purchasing power is

$$a^{\text{nom}} = (\nu_s)_{s=1}^{S} = \left(\frac{h(\omega_s)}{M_s}\right)_{s=1}^{S}$$

The real payoff is riskless only if $M_s$ is proportional to $h(\omega_s)$: to avoid introducing monetary risks in the nominal bonds traded in the private sector, the monetary authority would need to perfectly adjust the quantity of money to the (utility) value of the aggregate endowment $h(\omega_s)$.

If the bond is indexed to a bundle of goods $b$, i.e. if its nominal payoff is $p_s b$, then its real purchasing power is

$$a^{\text{ind}} = \nabla h(\omega_s)\, b$$

which is independent of the monetary policy, but depends on the variability of the relative prices $\nabla h(\omega_s)$. To avoid these relative-price fluctuations, the reference bundle would need to be state contingent and proportional to $\dfrac{\omega_s}{h(\omega_s)}$, since $\dfrac{\nabla h(\omega_s)\omega_s}{h(\omega_s)} = 1$ (see Geanakoplos-Shubik (1990) for a related analysis). Varying the reference bundle with the economic circumstances does not correspond to practice, since it would be costly to implement and would create a problem of credibility when the government calculating the index also has indexed payments or receipts. The assumption that the indexing bundle is non contingent seems realistic but it implies that indexing is not perfect.

Assuming that only one bond is introduced, Chapter 25 discusses the circumstances in which the nominal bond gives a higher social welfare than an indexed bond, concluding that, if the ppm does not vary too much and is negatively correlated with aggregate output—i.e. inflation is positively correlated with aggregate output — then the nominal bond is better than an imperfectly indexed bond. When inflation becomes too variable, relative-price risks become less important than inflation risks, and an indexed bond is preferred to a nominal bond.

A weakness of the analysis is that the comparison is made under the assumption that only one bond is traded, when nothing prevents both a nominal bond and an indexed bond from co-existing. The analysis has been extended to the case where the two bonds can be traded by Mukerji-Tallon (2004). They show that if agents have vague information about relative price risks and are ambiguity averse, for sufficiently low inflation the indexed bond will not be traded, even though both the nominal and the indexed bond are in the set of tradable securities.

In all the models considered so far, the payoff of a security is the same for buyers and sellers. In the last chapter Dubey-Geanakoplos-Shubik (DGS) (2005, Chapter 26) explore the role of default by allowing the amount delivered on securities to be tailored to the particular needs of the borrower. The paper also gives insight into the endogenous choice of security structure in equilibrium.

A financial contract is an exchange of current income for the promise to deliver income in the future. Such exchanges will function only if agents have incentives to deliver on their promises, or equivalently incur penalties for failing to pay their debts. Implicit in the models studied so far is the assumption that penalties for not paying on promises are so high that agents always fulfill the terms of their contracts. In practice penalties are not so high and some amount of default takes place. The DGS model adds explicit penalties for default to the GEI model and lets agents choose whether to keep their promises or to default on some of their commitments. Since it is a two-period model it cannot be too realistic in describing the way society enforces contracts, and directly postulates the existence of penalties which decrease the utility of the defaulting agent proportionally to the amount of default. For default to be compatible with equilibrium, it must be foreseen by the lenders whose plans must take into account the actual amounts that borrowers will deliver. The DGS model assumes that all contracts—which, as in the usual GEI model, have exogenously fixed terms—are run by intermediaries which pool the actual deliveries of the borrowers (sellers) on the contracts and pay the lenders (buyers) the same proportion of the promised delivery. Canonical examples of such contracts are securitized mortgages or credit-card debts, which are sold in pools, and for which investors receive the average rate of payment by the borrowers. Pooling is a way of keeping the markets anonymous and competitive: buyers only need to predict average payout rates which depend on aggregate characteristics of the borrowers.

A simple version of the model takes as its starting point a two-period GEI exchange economy with $J$ numeraire assets, each promising to deliver the amount $(V_s^j)_{s=1}^S \in \mathbb{R}_+^S$, $j = 1, \ldots, J$, of the numeraire good at date 1. The characteristics $(u^i, \gamma^i, \omega^i)$ of agent $i$ consist of a utility function $u^i : \mathbb{R}_+^{L(S+1)} \to \mathbb{R}$, (utility) default penalty rates $\gamma^i = (\gamma_s^{ji})_{s=1}^S \in \mathbb{R}_+^S$, $j = 1, \ldots, J$, where $\gamma_s^{ji}$ is agent $i$'s penalty for one unit of default on security $j$ in state $s$, and endowments $\omega^i \in \mathbb{R}_+^{L(S+1)}$. As usual agents can trade on spot and financial markets with prices $(p, q)$. It is now important however to distinguish between agent $i$'s buying and selling activity for each security. Let $z^i = (\phi^i, \theta^i) = (\phi_1^i, \ldots, \phi_J^i, \theta_1^i, \ldots, \theta_J^i)$ be agent $i$'s portfolio, where $\phi_j^i$ is the amount sold and $\theta_j^i$ is the amount purchased of security $j$. As a seller of asset $j$, agent $i$ in principle promises to deliver $V_s^j \phi_j^i$ in state $s$ but can instead choose to deliver only $D_s^{ji} = \widetilde{V}_s^{ji} \phi_j^i$, with $\widetilde{V}_s^{ji} \leq V_s^j$, at the cost of incurring the utility penalty $\gamma_s^{ji}(V_s^j - \widetilde{V}_s^{ji})\phi_j^i$. As a buyer of asset $j$, agent $i$ is in principle promised $V_s^j \theta_j^i$, but correctly anticipates that he will only receive $K_s^j V_s^j \theta_j^i$ where

$$K_s^j = \frac{\sum_{i=1}^{I} \widetilde{V}_s^{ji} \phi_j^i}{\sum_{i=1}^{I} V_s^j \phi_j^i} \tag{19}$$

is the actual delivery rate on the "pooled" contract $j$. Agent $i$'s problem is to choose consumption, portfolio and default rates to maximize utility net of default penalties

$$u^i(x^i) - \sum_{j=1}^{J} \sum_{s=1}^{S} (V_s^j - \widetilde{V}_s^{ji})\phi_j^i \tag{20}$$

given the date 0 and date 1 budget equations

$$
\begin{aligned}
p_0(x_0^i - \omega_0^i) &= -q(\theta^i - \phi^i) \\
p_s(x_s^i - \omega_s^i) &= \sum_{j=1}^{J} K_s^j V_s^j \theta_j^i - \sum_{j=1}^{J} \widetilde{V}_s^{ji} \phi_j^i, \qquad s = 1, \ldots, S
\end{aligned}
\tag{21}
$$

where the price of the numeraire good in which the assets deliver is normalized to 1 in each state. Note that choosing the sales and delivery rates $(\phi^i, \widetilde{V}_s^i)$ is equivalent to choosing the sales and delivery amounts $(\phi^i, D_s^i)$, where $D_s^{ji} = \widetilde{V}_s^{ji} \phi_j^i$, and the agent's maximum problem is convex in the choice variables $(x^i, \phi^i, \theta^i, D^i)$.

An equilibrium for the economy consists of actions, prices and delivery rates $((x, \phi, \theta, D), (p, q, K))$ such that (i) each agent maximizes (20) subject to (21), (ii) spot markets clear: $\sum_i (x^i - \omega^i) = 0$, (iii) security markets clear : $\sum_i (\phi^i - \theta^i) = 0$, and (iv) the anticipated delivery rates for assets traded in equilibrium coincide with the weighted average of the realized delivery rates (19). Since each agent's maximum problem is convex in the choice variables, an equilibrium can be shown to exist.

Actually the description of an equilibrium just given is incomplete since the delivery rates in (19) are not defined for non-traded assets with $\phi_j^i = 0$ for all $i$. An equilibrium is well defined only if all securities have prices and anticipated delivery rates which justify the trades chosen by the agents, including the absence of trade if all agents choose $\phi_j^i = \theta_j^i = 0$ on security $j$. Since rational expectations cannot contradict any expected delivery rate on a contract which is not traded, DGS consider a preliminary concept of equilibrium in which $K^j$ is defined by (19) if there is trade and is arbitrary if there is no trade.

However this definition yields many equilibria with no trade since, for any security $j$, a sufficiently low price $q_j$ will discourage sellers from using the security to borrow, while sufficiently low anticipated delivery rates $K_s^j$ will discourage buyers from using the contract to save. To eliminate such equilibria DGS introduce a refinement of equilibrium obtained as a limit of equilibria in which an outside agent (say the government) buys and sells $\epsilon$ of each security and fully delivers on its promises. This device eliminates excessively pessimistic expectations.

The DGS model permits a rich variety of questions on the role and consequences of default to be formalized and explored. For example they show that if the financial markets include Arrow securities with perfectly tailored state-contingent payoffs, then a Pareto optimal allocation is obtained by setting infinite penalties on these contracts and arbitrary penalties on any other contract, which will not be used in equilibrium. In short default has no role or justification when markets are complete. However if markets are incomplete—for example if the only security is the non-contingent bond for borrowing and lending—then default is useful since it permits agents to borrow even if there are "bad" states in which they are unable to repay. Intermediate penalties which ensure sufficient delivery rates to encourage savers to use the securities, while leaving room for some default by borrowers exposed to unusually bad draws, improve on the outcome of the standard GEI model with its (implicit) infinite penalties for default.

**Further Developments**

This volume has presented contributions to the theory of incomplete markets over a finite horizon—and most of the papers have focused on two-period economies. This setting provides the simplest version of the incomplete-market model: understanding the structure and properties of the model in this case is a first step toward exploring the consequences for an equilibrium of the fact that financial markets typically do not provide complete contingent contracts tailored to the risks and circumstances of each economic agent.

The simplifications obtained by focusing on the short horizon come however at a cost in realism. Two-period models do not lend themselves readily to calibration and numerical evaluation of the orders of magnitude of the effects revealed by the theoretical analysis. A related problem is that a two-period model tends to overestimate the cost of the incompleteness of markets, because it cannot incorporate the effect of dynamic trading strategies akin to carry-over strategies—saving when an income shock is favorable, dissaving or borrowing when it is unfavorable—which, in a stationary setting, can achieve almost perfect risk sharing with very few securities. More generally some equilibrium problems depend importantly on the fact that there is no natural terminal date at which the economy ceases, and need to be studied in a model with an open-ended future.

The study of incomplete market economies has thus progressively evolved from finite to infinite horizon economies, leading to a convergence between general equilibrium and macroeconomics. Exploring incomplete market economies over an open-ended future brings with it interesting new technical and conceptual issues, which will be the subject of the second volume on Incomplete

Markets.

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# Incomplete Markets

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