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Dynamic Processes for Tax Reform Theory*

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This paper presents a study of a procedure aimed at improving an indirect tax system. The procedure has the following features:

(A) The procedure operates in an idealized economic world analogous to that described by Diamond and Mirrlees in their basic model of [5]: All commodities can be taxed, producers have a competitive behavior, consumers only have a labor income. As it is supposed that there are no public goods, the tax system performs essentially a redistributive rather than a financing function.

—The principles of the procedure are borrowed from a preceding proposition of Guesnerie [7]: Commodity taxes are (infinitesimally) modified so that, simultaneously, market clearing conditions remain satisfied and the cost of consumption bundles of all households is decreased. Such changes in the tax system can be computed and define directions of tax reform inducing feasible and desirable moves in the economy.

(B) A dynamic process indexed by a continuous variable (time) is generated through the linkage of the desirable infinitesimal tax changes. This dynamic process is built in such a way that it has the following characteristics of feasibility and monotonicity:

· Feasibility: On every trajectory (if any) if one stops at any time t, the corresponding state of the economy is a feasible state (i.e., an equilibrium with respect to taxes prevailing at this time).

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· Monotonicity: The welfare of all households in the economy—as measured by some utility level—is an increasing function of the time variable on every trajectory.

(C) The specific purpose of the study of the dynamic process—once it has been defined—concerns the standard questions raised by dynamic systems, i.e., existence and stability. It is worth noting that the differential system considered has an inherent multivalued right-hand side and that the recent mathematical developments in the study of such systems provide us natural and appropriate tools (see Castaing [2], and Champsaur *et al.* [3]).

The reader will have noted that the preoccupations underlying this study have a narrow resemblance with those leading to the study of planning procedures concerning either the implementation of efficient allocation in the production sector, cf. [8], or the choice of efficient output levels for public goods [6], etc. This resemblance relies both on the definition of similar requirements for the dynamic processes—feasibility, monotonicity as defined by Malinvaud [11]—and, on the similarity of the questions raised: existence and stability.

The interpretation of the system as a tâtonnement procedure aimed at allowing the Center to implement efficient plans through an exchange of information with decentralized units, i.e., in the implicit framework of an economic theory of socialism (cf. Heal [8]) would suppose that the information gathered at any step by the Center concerns the individual consumption of households, elasticities of demand, and elasticities of supply.

However, a more natural interpretation of the procedure—and our assumptions are generally implicitly related to this view—consists in considering that the dynamic process describes the working of an algorithm used by the Government of a noncentrally planned economy, for revising its indirect tax system. Such an algorithm, which assumes the knowledge of demand and supply functions which may be provided by econometric estimates, would operate at some aggregate level (aggregation of commodities and households in classes).

1. MODEL AND NOTATION

1.A. *The Agents and the Assumptions*

The model we are considering was first explicitly introduced in the literature by Diamond and Mirrlees [5], who, in their seminal article, focused their attention on the derivation of optimality conditions for the tax system.

There are n commodities in the economy indexed by $k = 1 \cdots n$.

Two categories of commodities are considered. Commodities 1 to n_1 can only be consumed in negative quantities (or supplied) and commodities

$n_1 + 1$ to n can only be consumed in positive quantities (or demanded), this being true for all consumers. An appropriate choice of consumption sets allows taking into account this assumption of specialized commodities.

Let the consumers be indexed by $h = 1, \ldots H$, and let Ω_h be the consumption set of agent h. The following assumption on Ω_h is made:

H1. Ω_h is closed convex, bounded from below, and included in $\mathbb{R}_-^{n_1} \times \mathbb{R}_+^{n-n_1}$.[1]

Each consumer h has preferences defined on Ω_h and represented by a utility function u_h which satisfies:

H2. u_h is positive, strictly quasi-concave and continuously differentiable on $\mathrm{ri}(\Omega_h)$ with $\partial u_h/\partial x_k > 0$, $\forall h, \forall k$.

Faced with the price system π (π in $P = \mathring{\mathbb{R}}_+^n$), consumer h determines his demand by solving the program:

$$\mathrm{Max}\{u_h(x_h)/x_h \in \Omega_h,\ \pi \cdot x_h \leqslant 0\}.$$

The reader will notice that, through this formulation, the consumer is supposed to have no other income than his labor income.

From H1 and H2, this program has a unique solution $X_h(\pi)$ such that $\pi \cdot X_h(\pi) = 0$.

X_h is the demand function of consumer h.

$X = \sum_h X_h$ is the aggregate demand function.

We will also consider in the following the indirect utility function $V_h(\pi) = u_h(X_h(\pi))$. Production possibilities are described through a production function G (which defines the production set $Y = \{y/G(y) \leqslant 0\}$) such that:

H3α. G is strictly quasi-convex, and monotonic:

$$y > y' \Rightarrow G(y) > G(y'); \qquad G(0) = 0.$$

H3β. The asymptotic cone of Y, AY is $\mathbb{R}_-^n$.

Faced with the production price system p (in P), the producer determines his supply by solving the following program:

$$\mathrm{Max}\{p \cdot y, G(y) \leqslant 0\}.$$

It results from H3α and H3β that $\forall p \in P$, this program has a unique solution $\eta(p)$ such that $G(\eta(p)) = 0$. $\eta : P \to \mathbb{R}^n$ is the supply function.[2]

The supply and demand functions will be supposed to fulfill the following requirements:

[1] Unless we give an explicit contrary statement, the following conventions will hold thoughout the paper: consumption or production plans are column vectors; price systems are line vectors; $\bar{X}$, $\dot{X}$, ri X, Fr $\cdot$ X, denote, respectively, the closure, the interior, the relative interior, and the frontier of the set X; ${}^tA$ denotes the transpose of the matrix A.

[2] If G is strictly quasi-concave, and $A\mathring{Y} = \mathbb{R}_-^n$, it follows from Artzner and Neuefeind [1] that $BY =^{\mathrm{def}} \{p \mid \eta(p)$ is defined$\} = \mathring{\mathbb{R}}_+^n$.

H4. X_h is C^1 on P.

H5. η is C^1 on P.[3]

Given H4 and H5, one will denote by $\partial\bar{X}(\pi)$ the $n \times n$ matrix

$$\partial\bar{X}(\pi) = l\left(\cdots \frac{\partial X^l}{\partial \pi_k}(\pi) \cdots\right) \quad (\text{row } l,\ \text{column } k)$$

and by $\partial\bar{\eta}(p)$ the $n \times n$ matrix:

$$\partial\bar{\eta}(p) = l\left(\cdots \frac{\partial \eta^l}{\partial p_k}(p) \cdots\right).$$

It is well known that : $p \cdot \partial\bar{\eta}(p) = 0$. So that $\partial\bar{\eta}(p)$ is at most of rank $n - 1$. We shall assume precisely that:

H6. $\partial\bar{\eta}(p)$ is of rank $(n - 1)$, $\forall p \in P$.

H7. There does not exist $(\pi, p) \in P \times P$ such that:

$$X(\pi) = \eta(p), \qquad p \cdot \partial\bar{X}(\pi) = 0.$$

This is a kind of regularity assumption analogous to the Kuhn–Tucker-type regularity condition.

This property does not look very restrictive and it is argued[4] that it is likely to be "generic" in the sense of differential topology.

H7′ is more restrictive:

H7′. There does not exist $(p, \pi) \in P \times P$ such that

$$X(\pi) \leqslant \eta(p), \qquad p \cdot \partial\bar{X}(\pi) = 0.$$

Now let $V(p) = \{u \in \mathbb{R}^n \mid p \cdot u = 0\}$.

From a geometric point of view, $V(p)$ which is normal to p is a hyperplane parallel to the tangent hyperplane to the production set Y at the point $\eta(p)$. From assumption H6, it can be proved that the image of the linear mapping $\partial\bar{\eta}(p)$ is $V(p)$. Actually, it follows from Guesnerie [7, Lemma 1] that if one denotes by $\partial\tilde{\eta}(p)$ the restriction of $\partial\bar{\eta}(p)$ to $V(p)$, $\partial\tilde{\eta}(p)$ is a

[3] The assumption that $\mathbb{R}_-^n$ is the asymptotic cone of Y does not play a decisive role in the following. In fact, it would be enough to consider throughout this paper that η, which is from $H3\alpha$ and Artzner and Neuefeind [1], defined on the interior of the polar of AY, is differentiable in this set.

[4] If one considers the set of equations $X(\pi) = \eta(p)$, $p \cdot \partial\bar{X}(\pi) = 0$, one sees that we have $2n - 1$ equations for $2n - 2$ variables. Hence, this suggests that they can only be satisfied for exceptional data, a point which could be confirmed by a more formal approach.

one-to-one correspondence from $V(p)$ onto $V(p)$. The inverse of $\partial\tilde{\eta}$ will be denoted in the following $\partial\tilde{\eta}^{-1}(p)$.

Intuitively, this property means that any (infinitesimal) supply variation lying in $V(p)$ (i.e., normal to p) can be obtained through a variation of production prices normal to the initial price system (which is a normalization constraint) and conversely.

Let us briefly discuss the restriction in the range of production possibility sets implied by H3α–H3β–H5–H6. One may first remark that given H3α and β, H5 and H6 are mainly technical and it is likely that it could be proved that they are "generic" properties. H3β supposes some substitutability between inputs and outputs and that marginal returns tend to vanish when the scale of production tends to infinity. On the one hand, it is not unreasonable at the aggregate level we are considering; on the other hand, this assumption is not necessary for the argument and is only intended to simplify the presentation.

H3α, even if it is not unusual in the literature, is perhaps less satisfactory since it rules out production sets where inputs and outputs can be completely distinguished. Actually H3α could be relaxed for allowing such production sets, but condition (A) below should then be reinforced.[5]

H3α seemed an acceptable compromise between realism and simplicity.

Finally, a last assumption will be made which concerns demand behavior:

ASSUMPTION (A). (α) $X_h(\pi) \in \mathrm{ri}(\Omega_h)$, $\forall \pi \in P$.

(β) If a sequence (π_n) of vectors in P is such that:

$$\|\pi_n\| = \|\pi_0\| \qquad \forall n, \|\pi_0\| \neq 0;^{6}$$

$$\pi_n \xrightarrow[n\to+\infty]{} \bar{\pi} \in \bar{P}/P.$$

Then,

(a) either $\|X(\pi_n)\| \to +\infty$,

(b) or $\forall v > 0$, $\exists h \in (1 \cdots H)$ and N such that $n > N \Rightarrow u_h(X_h(\pi_n)) < v$.

The idea underlying Assumption (A) is that when a commodity price tends to zero, either it is a consumption good and its demand tends to infinity (a) or it is a labor-type commodity and then the utility level of some household tends to zero (b).

Relative to H1–H6, (A) introduces additional restrictions on the preferences which are considered. For example, the reader will check that (A) will hold if all consumption commodities are supposed essential ($x_{ih} = 0 \Rightarrow u_h(x_h) = 0$, $i = 1 \cdots n_1$, $h = 1 \cdots H$) and if each consumer is specialized (he can supply one and only one type of labor).

This brief analysis of Assumption (A) suggests that it can be considered rather strong if one reasons at the disaggregate level. However, at a more aggregate level, which is an appropriate level for practical use of the algorithm proposed in the note, the assumption does not look unreasonable.

[5] It should assure that: (1) If the price of one output tends to zero its supply tends to zero. (2) All commodities are "essential". With $H3\alpha$ the things are simpler since the supply of one output tends to $-\infty$ when its price tends to zero.

[6] $\|x\|$ is the euclidean norm of x.

1. B. *Equilibria and the Principles of an Algorithm for Tax Reform*

Let us give several definitions:

DEFINITION 1. An *equilibrium* of the system consists in a couple (π, p) of consumption price system and production price system such that:

$$\sum_{h=1}^{H} X_h(\pi) \leqslant \eta(p).$$

DEFINITION 2. The equilibrium is said to be *tight* if $\sum_{h=1}^{H} X_h(\pi) = \eta(p)$. If the latter equality does not hold, the equilibrium is *nontight* or *inefficient*.

Definition 1 describes an equilibrium with taxes: the disconnection between production and consumption price systems is supposed to be implemented through taxation. The formulation also supposes that the profit of the firms is completely taxed by the government. (For a comprehensive discussion of these assumptions see Diamond and Mirrlees [5].)

Definition 2 expresses a tightness condition which is the equivalent for our model of the efficiency property of Diamond and Mirrlees. If this condition is not satisfied, the total demand could be satisfied with an inefficient production plan, i.e., a production plan in the interior of the total production set.

In this note, one will particularly be interested in special types of equilibria, the Pareto equilibria, defined as follows:

DEFINITION 3. A *local Pareto equilibrium* consists in a couple (Π, p) such that:

(α) (π, p) defines a tight equilibrium;

(β) $-p \cdot \partial\bar{X}(\pi) = {}^t(\sum_{h=1}^{H} \lambda_h X_h(\pi))$ with $\lambda_h \geqslant 0$.

Conditions (β) are necessary conditions for second best Pareto optimality (see Diamond and Mirrlees [5]) and this fact justifies the vocabulary of local Pareto equilibria.

Actually, a local Pareto equilibrium may have different features:

· It may be a *global second best Pareto optimum* in the sense that there does not exist, in the set of all equilibria, one Pareto superior equilibrium.

Moreover, under the conditions we consider, all Pareto optima belong to the set of local Pareto equilibria.

· It may be a *local Pareto optimum* in the sense that there does not exist a neighbor Pareto superior equilibrium.

· It may be neither a local nor a global second best Pareto optimum.

In this last category fall the *saddle-type Pareto equilibria*, to which we will come back later.

We shall now briefly present the algorithm for tax reform we want to consider. For more details the reader is invited to refer to Guesnerie's paper [7] where the principles of the algorithm have first been presented and discussed.

Starting from price systems $p(t)$ and $\pi(t)$ for consumption and production at date t, the algorithm computes (infinitesimal) changes dp and $d\pi$ in order to meet the two following criteria:

—These changes should be feasible, all markets remaining cleared.

—They should increase the welfare of all consumers (Pareto improvement conditions).

With the notation of [7], let us introduce the following sets:

$$\mathring{K}(\pi) = \{a \in \mathbb{R}^n \mid a \cdot X_h(\pi) < 0,\ h = 1 \cdots H\};$$

$$Q(\pi, p) = \{a \in \mathbb{R}^n \mid p \cdot \partial\overline{X}(\pi) \cdot {}^t a \leqslant 0\};$$

$$\operatorname{Fr} Q(\pi, p) = \{a \in \mathbb{R}^n \mid p \cdot \partial\overline{X}(\pi) \cdot {}^t a = 0\}.$$

The algorithm links infinitesimal changes of taxes from a given starting equilibrium, according to the following conditions:

$$\frac{d\pi}{dt} \in \mathring{K}(\pi) \cap \operatorname{Fr}\{Q(\pi, p)\}; \tag{1a}$$

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p) \cdot \partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt}. \tag{1b}$$

An intuitive understanding of the above system can be gained through the following remarks:

· The change in consumption prices is first constrained to be in Fr $Q(\pi, p)$, i.e., to induce a change in demand $\partial\overline{X}(\pi) \cdot {}^t d\pi/dt$ whose value expressed with production prices is zero: $(p \cdot \partial\overline{X}(\pi) \cdot {}^t d\pi/dt = 0)$.

· It follows that the change in demand induced by the change in consumption prices belongs to $V(p)$ and can then be matched by a change of production prices defined by formula (1b) (since $\partial\tilde{\eta}^{-1}(p)$ is defined on $V(p)$).

· The change in consumption prices is such that the value of all consumption bundles decreases $(d\pi/dt \in \mathring{K}(\pi))$, a condition aiming at Pareto improvement.

More precisely, the interest of system (1) lies in the following precise property (which is Corollary III in Guesnerie [7]).

PROPOSITION 1. *If there exists a solution of system* (1) $(\pi(t), p(t))$ *defined on* $[0, T]$ *and starting from* $(\pi(0), p(0))$ *such that* $\eta(p(0)) = X(\pi(0))$, *then*

$(X_h(\pi(t)), \eta(p(t)), \pi(t), p(t))$ *define a tight equilibrium* $\forall t \in [0, T]$,

$V_h(\pi(t))$ is a strictly increasing function of t.

The rest of the paper is devoted to a comprehensive study of dynamical processes governed by system (1) or by similar systems. This study raises the types of questions that are usually considered in the literature on planning procedures (Heal [8], Malinvaud [11]) and which are twofold:

(1) Do there exist solutions of the system of differential equations which make it meaningful?

(2) Given a solution path of the system, does it converge and to which points?

These two problems—existence problems and convergence problems—will be examined in Section 2 for a system derived from system (1). It will be shown that some limit points of solution paths have undesirable properties. A more complicated system which excludes such undesirable limit points will then be considered in Section 3. All proofs are in the Appendixes.

2. A First Dynamical Process

System (1) will be slightly modified in order to be defined for all $(\Pi, p) \in P \times P$. Thus, one will assume that prices remain constant as soon as the second member of Eq. (1a) becomes empty. This gives system (1′)

$$\begin{aligned} \frac{d\pi}{dt} &\in \mathring{K}(\pi) \cap \operatorname{Fr} Q(\pi, p) && \text{if } \mathring{K}(\pi) \cap \operatorname{Fr} Q(\pi, p) \neq \varnothing \\ &= 0 && \text{otherwise,} \\ \frac{{}^t dp}{dt} &= \partial\tilde{\eta}^{-1}(p) \cdot \partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt}. \end{aligned} \tag{1′}$$

2.A. *Existence Problems*

Has system (1′), which is a system of multivalued differential equations, a solution?

The answer to this question rests on mathematical results sometimes directly initiated by problems in economic theory (cf. Henry [10]). A view of these results is given in Champsaur *et al.* [3].

Here, a way of approaching existence is to consider an auxiliary system obtained by building an upper-hemi-continuous convex compact valued correspondence M which is extracted from $\mathring{K}(\pi) \cap \operatorname{Fr} Q(\pi, p)$ when it is nonempty.

To this end let us consider:

$$\varphi(\pi, p) = \{a \in \mathbb{R}^n / p \cdot \partial \overline{X}(\pi) \cdot {}^t a = 0, \pi \cdot {}^t a = 0, \| a \| \leqslant 1\};$$

$$f(\pi, p) = \operatorname{Max}\{\operatorname{Min}_h(-a \cdot X_h(\pi))/a \in \varphi(\pi, p)\}.$$

Let us notice that $f(\pi, p) = 0$ if and only if $\mathring{K}(\pi) \cap \operatorname{Fr} Q(\pi, p) = \varnothing$, $M(\pi, p) = \{a \in \varphi(\pi, p)/f(\pi, p) = \operatorname{Min}_h(-a \cdot X_h(\pi))\}$.

System (2) is:

$$\frac{d\pi}{dt} \in M(\pi, p);$$
$$\frac{{}^t dp}{dt} = \partial \tilde{\eta}^{-1}(p) \cdot \partial \overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt}. \tag{2}$$

From an economic point of view, system (2) has the following characteristics:

· the speed of change of consumption prices is bounded: $\| a \| \leqslant 1$;

· the norm of the consumption price system remains constant along any solution path (as does the norm of p);

· the changes in prices are designed in order to maximize the smallest speed of decrease in the agent's consumption bundle value.

The following theorem holds:

THEOREM 1. *Under assumptions* H1–H7 *and* (A), *for all* $(\pi^0, p^0) \in P \times P$ *such that* $\eta(p^0) = X(\pi^0)$, *there exists a solution* $(\pi(t), p(t))$ *of system* (2), *defined on* $[0, +\infty[$ *and starting from* (π^0, p^0).

COROLLARY 1. *The same statement is true for system* (1′).

In other words, in the terminology of general dynamical processes there exists a trajectory for systems (1′) and (2). Let us recall that according to Proposition 1, along this trajectory utilities increase and equilibria remain tight.

A solution $(\pi(t), p(t))$ on $[0, +\infty[$ of system (2), if any, allows us to define a solution $(\tilde{\pi}(t), \tilde{p}(t))$ of system (1′) on $[0, +\infty[$ in the following way: From $t = 0$ until the first time T where $f(\pi(T), p(T)) = 0$, $\tilde{\pi}(t) = \pi(t)$, $\tilde{p}(t) = p(t)$, $t \in [0, T]$; for $t > T$, $\tilde{\pi}(t) = \tilde{\pi}(T)$, $\tilde{p}(t) = \tilde{p}(T)$.[7]

This allows us to prove Corollary 1.

2.B. *Quasi-stability*

Let us recall first some definitions relative to a general dynamical process (P) governed by the system of multivalued differential equations:

$$\frac{dx}{dt} \in F(x); \tag{S}$$

$\bar{x}$ is an *equilibrium* of (P) if $0 \in F(\bar{x})$.

A *trajectory* of (P) is a solution of (S) defined on $[0, +\infty[$.

We will say that $\bar{x}$ is a *limit point* of a trajectory $x(t)$ if there exists a sequence $t_n \to_{n \nearrow +\infty} +\infty$ such that : $x(t_n) \to_{n \nearrow +\infty} \bar{x}$.

Process (P) is *quasi-stable* iff any limit point of a trajectory is an equilibrium.

For the systems we consider, the following propositions, which are proved in the Appendixes, hold:

PROPOSITION 2. *System* (2) *is quasi-stable.*

THEOREM 2. *For any trajectory of system* (1′) *such that*: $\forall h (1 \cdots H)$ $\forall t \geqslant 0$, $-d\pi/dt \cdot X_h(\pi(t)) \geqslant k f(\pi(t), p(t))$, *where* k *is a strictly positive number* (*smaller than one*), *every limit point is an equilibrium.*

Remark. An obvious requirement for a limit point of a trajectory to be an equilibrium for systems like (1′) and (2) is that the speed of price changes does not tend to zero "too fast" on the trajectory.

This requirement is automatically satisfied on a trajectory of system (2)—which then can be proved quasi-stable—but it is clear that it could be violated on some trajectory of system (1′), independently of the primitive characteristics of the economy. (This could occur, for example, with a speed of adjustment such that $d\pi/dt$ tends exponentially fast to zero with time). So the condition given in Theorem 2 is a (weak) sufficient condition ensuring that the speed of price changes does not become too small and excluding the trajectories when this would happen (trajectories which are rather uninteresting from a Government or Planner point of view).

A similar condition, calling for similar remarks, will be used later in Theorem 4.

[7] In system (2), we obtained the upper-hemi-continuity of M when it passes through 0, to the cost of introducing vectors $d\Pi/dt$ which were not allowed by system (1′).

Let us now consider a trajectory of system (1′) where the speed meets the requirement $-d\pi/dt \cdot X_h(\pi(t)) \geqslant k(f(\pi(t), p(t))$; such a trajectory exists from Theorem 1. Let $(\bar{\pi}, \bar{p})$ be a limit point of this trajectory. According to Theorem 2, $(\bar{\pi}, \bar{p})$ is an equilibrium of system (1′), i.e.,

$$\mathring{K}(\bar{\pi}) \cap \operatorname{Fr} Q(\bar{\pi}, \bar{p}) = \varnothing.$$

What can be said about such an equilibrium?

An answer is provided by Proposition 3, which can be seen as a corollary of Proposition 4 in Guesnerie [7].

PROPOSITION 3. *If $(\bar{\pi}, \bar{p})$ is a limit point of system* (1′)

—either it is a local Pareto equilibrium;
—or $\mathring{K}(\bar{\pi}) \cap Q(\bar{\pi}, \bar{p}) \neq \varnothing$.

Thus, we are faced with four types of possible limit points:

(α) $(\bar{\pi}, \bar{p})$ is a global second best optimum;

(β) $(\bar{\pi}, \bar{p})$ is a local Pareto optimum;

(γ) $(\bar{\pi}, \bar{p})$ is a saddle-type Pareto equilibrium. It is such that there exist Pareto superior points in any neighborhood of $(\bar{\pi}, \bar{p})$ but, however, the necessary conditions of second best Pareto optimality are satisfied. Our algorithm stops in these points because it does not allow, even temporarily, a null speed of increase of one agent's utility.

(δ) If $\mathring{K}(\bar{\pi}) \cap Q(\bar{\pi}, \bar{p}) \neq \varnothing$, $(\bar{\pi}, \bar{p})$ is not a local Pareto equilibrium in the sense of Definition 3. And generally there exist Pareto superior equilibria in any neighborhood of $(\bar{\pi}, \bar{p})$, but they are (or may be) nontight so that the process associated to system (1′), which is constrained to remain in the set of tight equilibria, stops. With the vocabulary of Guesnerie [7] in which this phenomenon—which may look strange—has been studied, there exist strictly Pareto improving directions of price changes but they lead (or at least tend to lead) to nontight equilibria. In other words a time path of price changes inducing a monotonic increase of all utilities can be extended only if temporary[8] inefficiencies are allowed, which is not the case for systems (1), (1′), and (2).

Limit points corresponding to (δ) are particularly unsatisfactory. We would try to rule them out, by considering in Section 3 a more complicated system which will allow temporary inefficiencies.

[8] Inefficiencies are only temporary, in the sense that the attainment of a global second best optimum would remove them.

3. A DYNAMIC PROCESS WITH TEMPORARY INEFFICIENCIES

Defining $Q(\pi, p, \lambda) = \{a \in \mathbb{R}^n / p \cdot \partial\overline{X}(\pi) \cdot {}^t a \leqslant \lambda \| p \|\}$ where λ is a parameter belonging to $\mathbb{R}$, we will be interested in this section by the following system:

$$\begin{aligned}
&\frac{d\pi}{dt} \in \mathring{K}(\pi) \cap Q(\pi, p, \lambda) \qquad \text{if this set is not empty;}\\
&\frac{d\pi}{dt} = 0 \qquad \text{otherwise;}\\
&\frac{{}^t dp}{dt} = \partial\bar{\eta}^{-1}(p)\left[\partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt} + \frac{d\lambda}{dt}\,{}^t\pi + \lambda\frac{{}^t d\pi}{dt}\right]; \qquad (3)\\
&\frac{d\lambda}{dt} = -\frac{p \cdot \partial\overline{X}(\pi) + \lambda p}{p \cdot {}^t\pi} \cdot \frac{{}^t d\pi}{dt};\\
&\left\| \frac{d\pi}{dt} \right\| \leqslant 1.
\end{aligned}$$

Solutions of this system (if any), have the monotonicity property that we expect and possibly display temporary inefficiencies, as stated in

PROPOSITION 4.[9] *If there exists $(\pi(t), p(t), \lambda(t))$, a solution of system* (3) *starting from $(\pi(0), p(0), \lambda(0))$ and defined on $[0, T]$ with $\lambda(0)\,{}^t\pi(0) = \eta(p(0)) - X(\pi(0))$ and $\lambda(0) \geqslant 0$ then:*

· $\lambda(t) \geqslant 0, \forall t \in [0, T]$;

· $X(\pi(t)) + \lambda(t)^t\,\pi(t) = \eta(p(t)), \forall t \in [0, T]$;

· *$V_h(\pi(t))$ is a strictly increasing function of t, $\forall h = 1 \cdots H$, for all t where $\mathring{K}(\pi(t)) \cap Q(\pi(t), p(t), \lambda(t)) \neq \varnothing$.*

The proof of Proposition 4 is given in the Appendix and rests upon the fact that $\lambda(t) \leqslant 0$ would imply $(d\lambda/dt)(t) \geqslant 0$ (so that $\lambda(t)$ cannot become negative).

As in Section 2 we introduce an auxiliary system. Let

$$\varphi(\pi, p, \lambda) = \left\{\begin{array}{r} a \in \mathbb{R}^n\colon\ p \cdot \partial\overline{X}(\pi) \cdot {}^t a \leqslant \lambda \| p \|,\\ \pi \cdot {}^t a = 0,\\ \| a \| \leqslant 1. \end{array}\right\},$$

$$f(\pi, p, \lambda) = \max\{\min_h(-a \cdot X_h(\pi)),\ a \in \varphi(\pi, p, \lambda)\},$$

$$M(\pi, p, \lambda) = \{a \in \varphi(\pi, p, \lambda)\colon f(\pi, p, \lambda) = \min_h(-a \cdot X_h(\pi))\}.$$

[9] The reader will fruitfully compare this proposition with Corollary 3 in Guesnerie [7].

System (4) is the following:

$$\frac{d\pi}{dt} \in M(\pi, p, \lambda);$$

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p) \left[\partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt} + \frac{d\lambda}{dt}\, {}^t\pi + \lambda \frac{{}^t d\pi}{dt}\right]; \qquad (4)$$

$$\frac{d\lambda}{dt} = - \frac{p \cdot \partial\overline{X}(\pi) + \lambda p}{p \cdot {}^t\pi} \cdot \frac{{}^t d\pi}{dt}.$$

Theorems similar to those of Section 2 can be stated:

THEOREM 3. *Under Assumptions* H1–H6–H7′ *and* (A) *for all* $(\pi(0)\ p(0)) \in P \times P$ *such that* $\eta(p(0)) = X(\pi(0))$, *there exists a solution defined on* $[0, +\infty[$ *starting from* $(\pi(0), p(0), 0)$ *for systems* (3) *and* (4).

THEOREM 4. *System* (4) *is quasi-stable.*

For any trajectory of (3) *such that*: $\forall h(1 \cdots H), \forall t \geqslant 0, -d\pi/dt \cdot X_h(\pi(t)) \geqslant kf(\pi(t), p(t), \lambda(t)), k \in]0, 1]$, *every limit point is an equilibrium.*

COROLLARY 2. *For any trajectory of* (3) *meeting the above speed condition, every limit point is a local Pareto equilibrium.*

Thus with the vocabulary of Section 2, we proved that the limit points of a trajectory of (3) meeting the speed requirement are either second best Pareto optima (α) or local second best Pareto optima (β) or saddle-type Pareto equilibria (γ).

Obviously, one would wish to design process for which limit points fall in case (α). It is clear from the basic nonconvexity of the set of equilibria that such a property cannot be expected for processes which only consider local information on the feasible states.

If limit points of type (β) cannot be excluded, can one at least rule out case (γ) which is particularly unsatisfactory, by defining an appropriate process?

In the state of the art, it does not seem clear that such processes can be designed without looking at second order conditions, but further considerations on this point are outside the scope of this paper.

APPENDIX I

1

Proof of Proposition 1

$(\pi(t), p(t))$ being a solution of system (1), one has:

$$\frac{{}^t dp}{dt} = \partial\tilde{\eta}^{-1}(p) \cdot \partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt},$$

hence

$$\partial\bar{\eta}(p) \cdot \frac{{}^t dp}{dt} = \partial\overline{X}(\pi) \cdot \frac{{}^t d\pi}{dt}.$$

And thus $\forall t \in [0, T]$, $\eta(p(t)) = X(\pi(t))$ (because of the initial condition); i.e., the equilibrium remains tight. If V_h is the indirect utility function:

$$\frac{dV_h(\pi(t))}{dt} = \sum_{k=1}^{n} \frac{\partial V_h}{\partial \pi_k} \frac{d\pi_k}{dt}$$

and a classical calculus shows that there exist $\alpha_h > 0$, $h = 1 \cdots H$, such that:

$$\frac{dV_h(\pi(t))}{dt} = \sum_{k=1}^{n} - \alpha_h(\pi(t))\, X_h^k(\pi(t)) \frac{d\pi_k}{dt}$$

$$= -\alpha_h(\pi(t)) \cdot X_h(\pi(t)) \cdot \frac{{}^t d\pi}{dt},$$

but: $\forall h\ (1 \cdots H)$, $X_h(\pi(t)) \cdot {}^t d\pi/dt < 0$, and thus V_h is strictly increasing.

2

Proof of Theorem 1

The proof has two steps: In the first step, we prove that there is a local solution; in the second step, the solution is extended to $[0, +\infty[$.

Step 1. System (2) can be written $(d\pi/dt, {}^t dp/dt) \in F(\pi, p)$ where $F(\pi, p) = \{(a, \partial\tilde{\eta}^{-1}(p) \cdot \partial\overline{X}(\pi) \cdot {}^t a), a \in M(\Pi, p)\}$.

To prove local existence, we will refer to Castaing's theorem stated below (see Appendix 2), which requires that F be a compact convex valued upper-hemi-continuous (*uhc*) correspondence and be adequately bounded.

Let

$$T = \{(\pi, p) \in P \times P \mid p \cdot \partial\overline{X}(\pi) = 0\},$$

$$S = P \times P \backslash T.$$

One can prove:

(1) F is a compact convex valued upper-hemi-continuous correspondence on S. We note that φ is continuous (it is upper-hemi-continuous as an intersection of upper-hemi-continuous correspondences, and lower-hemi-continuous by an ad hoc argument (cf. Proposition 5 in Appendix 2)) and that $\text{Min}_h\{-a \cdot X_h(\pi)\}$ is a continuous function in (π, p, a). The maximum theorem then implies that M is upper-hemi-continuous and compact valued.

Let

$g : (\Pi, p) \in S \to \partial \bar{\eta}^{-1}(p) \cdot \partial \bar{X}(\Pi)$;

$\Phi : (x, A) \in \mathbb{R}^n \times \mathscr{L}(\mathbb{R}^n) \to (x, A\, {}^t x)$(where $\mathscr{L}(\mathbb{R}^n)$ is the set of all linear functions from $\mathbb{R}^n$ to $\mathbb{R}^n$).

g and Φ are continuous and $F = \Phi \circ (M, g)$ implies that F is upper-hemi-continuous and compact valued.

The fact that F is convex valued results from the concavity of the maximized function $\mathrm{Min}_h\{-a \cdot X_h(\Pi)\}$.

(2) F is bounded on any compact set K of S, as a consequence of the upper-hemi-continuity of F.

Hence, from Castaing's theorem one can infer that:

For all compact $K \subset S$ there exists $T_K > 0$, such that for all $(\Pi^0, p^0) \in K$, there is a solution $(\Pi(t)\, p(t))$ of system (2) starting from (Π^0, p^0) and defined on $[0, T_K]$.[10] (Take an open set containing the compact K and apply the theorem.)

Step 2. Let (Π^0, p^0) be such that $\eta(p^0) = X(\Pi^0)$ and let us consider a nondecreasing sequence of compact sets C^k such that $S = \bigcup_k C^k$.

Let $C^{k(0)}$ be the smallest compact set of the family containing $(\Pi^0, p^0) \cdot \exists T_{k(0)} > 0$ and $(\Pi_0(t), p_0(t))$ a solution of system (2) starting from (Π^0, p^0) and defined on $[0, T_{k(0)}]$.

Let $\Pi_0(T_{k(0)}) = \Pi^1, p_0(T_{k(0)}) = p^1$, and let $C^{k(1)}$ be the smallest compact set of the family containing (Π^1, p^1), etc.

Thus we build sequences $C^{k(n)}$, $T_{k(n)}$, $\Pi_n(t), p_n(t)$ such that $C^{k(n)}$ is the smallest compact set containing $(\Pi_{n-1}(T_{k(n-1)}), p_{n-1}(T_{k(n-1)}))$, and $(\Pi_n(t), p_n(t))$ is a solution path of system (2) defined on $[0, T_{k(n)}]$ and starting from $\Pi_{n-1}(T_{k(n-1)}), p_{n-1}(T_{k(n-1)})$.

We then have a solution on $[0, \sum_n T_{k(n)}]$ starting from Π^0, p^0 and such that:

(1) $\| p_n(t) \| = \| p^0 \|, \| \Pi_n(t) \| = \| \Pi^0 \|$,

(2) $X(\Pi_n(t)) = \eta(p_n(t))$ (from Proposition 1).

It remains to prove that $\sum T_{k(n)}$ is a divergent series.

From $(\Pi_n(T_{k(n)}), p_n(T_{k(n)}))$ one can extract a subsequence (Π_n, p_n) converging to $(\bar{\Pi}, \bar{p})$.

We will prove that $(\bar{\Pi}, \bar{p}) \notin \bar{S} \setminus S$.

If $p_n \to \bar{p} \in \bar{P}$, it would follow from H3 that $\| \eta(p_n) \| \to +\infty$ (cf. Artzner and Neuefeind [1, Theorem 1]). But H1–H3 imply that the set of tight equilibria of the model is contained in a compact set (cf. Debreu [4]) which leads to a contradiction.

[10] One must notice that the theorem gives more than a local existence statement which alone could be obtained by more elementary methods (cf. Guesnerie [7]) but would be insufficient for the following.

If $\pi_n \to \bar{\pi} \in \bar{P}$, Assumption (A) would imply either that $\| X(\pi_n) \| \to +\infty$, which is impossible for the reason just stated, or $U_h(X_h(\pi_n)) \to 0$ for some h, which contradicts the strict monotonicity of the process in terms of utility.

If $(\pi_n, p_n) \to (\bar{\pi}, \bar{p}) \in T$, $(X(\pi_n), \eta(p_n)) \to (X(\bar{\pi}), \eta(\bar{p}))$ such that $X(\bar{\pi}) = \eta(\bar{p})$ which contradicts H7.

Hence, there is a compact set $C^{\bar{k}}$ containing an infinity of points of the sequence $(\pi_n(T_{k(n)}), p_n(T_{k(n)}))$.

Let $\bar{T} = \mathrm{Min}(T^0, \ldots, T^{\bar{k}}) > 0$ which ensures that for an infinity of n: $T_{k(n)} > \bar{T}$. This concludes the proof of this step.

Proof of Proposition 2 *and Theorem* 2

Step 1. Let $E(k_1, k_2, r_1, \ldots, r_H) = \{(\pi, p) \in P \times P, \| \pi \| = k_1, \| p \| = k_2, \eta(p) - X(\pi) = 0, U_h(X_h(\pi)) \geqslant r_h, r_h > 0, \forall h\}$.

Let us consider a sequence (π_n, p_n) in E. One can extract a subsequence converging to $(\bar{\pi}, \bar{p})$ (because of the boundedness of the norm). An argument similar to that of Step 2 above shows that $(\bar{\pi}, \bar{p}) \in P \times P$ and hence, by the continuity of η, X, U_h, to E. Hence E is compact.

Let now $(\pi(t), p(t))$ be a trajectory of system (2), starting from $(\pi^0, p^0) \in P \times P$. Let (π^*, p^*) be a limit point. (Π^*, p^*) is an equilibrium if and only if:

$$0 \in F(\pi^*, p^*) \Leftrightarrow 0 \in M(\pi^*, p^*) \Leftrightarrow f(\pi^*, p^*) = 0.$$

For proving the statement, we shall show that $f(\pi^*, p^*) > 0$ is impossible.

Step 2. Let $f(\pi^*, p^*) = \epsilon$ be strictly positive. An easy argument shows that from a time t_0 on, there exist $r_h > 0$, such that the trajectory lies in the compact set $E(\| \pi^0 \|, \| p^0 \|, (r_h)) \stackrel{\text{def}}{=} K$, on which f is uniformly continuous. Furthermore $(\pi^*, p^*) \in K$.

The continuity of f implies:

$$\exists \eta > 0: \qquad \| (\pi, p) - (\pi^*, p^*) \| < \eta \Rightarrow f(\pi, p) > \epsilon/2.$$

Let $(\pi(t_n), p(t_n))$ be a sequence of points of the trajectory converging to (π^*, p^*), when $t_n \to \infty$

$$\exists n_0: \qquad \forall n > n_0, \qquad \| (\pi(t_n), p(t_n)) - (\pi^*, p^*) \| < \eta.$$

Hence: $f(\pi(t_n), p(t_n)) > \epsilon/2$ for $n > n_0$.

On the other hand, f being uniformly continuous on the trajectory $(\pi(t), p(t))$ (for $t \geqslant t_0$)

$$\exists \delta > 0: \qquad \forall n \geqslant n_0,$$
$$\| (\pi(t), p(t)) - (\pi(t_n), p(t_n)) \| < \delta \Rightarrow f(\pi(t), p(t)) > \epsilon/4,$$

$$\|\pi(t)-\pi(t_n)\| = \left\| \int_{t_n}^{t} \frac{d\pi}{d\tau}\, d\tau \right\| \leqslant \left| \int_{t_n}^{t} \left\| \frac{d\pi}{d\tau} \right\| d\tau \right| \leqslant |t-t_n|,$$

$$\|p(t)-p(t_n)\| = \left\| \int_{t_n}^{t} \frac{dp}{d\tau}\, d\tau \right\| \leqslant \left| \int_{t_n}^{t} \left\| \frac{dp}{d\tau} \right\| d\tau \right|.$$

As ${}^t dp/dt = d\bar{\eta}^{-1}(p(t))\, \partial\bar{X}(\pi(t))\, {}^t d\pi/dt$ and $\partial\bar{\eta}^{-1}(p(t)) \cdot \partial\bar{X}(\pi(t))$ is continuous on K and hence bounded, there exists k such that:

$$\left\| \frac{dp}{d\tau} \right\| \leqslant k \qquad \forall\tau \in [t_n, t] \Rightarrow \|p(t)-p(t_n)\| \leqslant k\,|t-t_n|.$$

It follows that: $|t-t_n| < \delta/(1+k^2)^{1/2} = \mu$ implies:

$$\|(\pi(t), p(t)) - (\pi(t_n), p(t_n))\| < \delta$$

and hence $f(\pi(t), p(t)) > \epsilon/4$.

Now the function $V_h{}^*(t) = U_h(X_h(\pi(t))$ is continuous nondecreasing,

$$\Rightarrow V_h{}^*(t) \xrightarrow[t \nearrow +\infty]{} V_h{}^* = U_h(X_h(\pi^*)) \qquad \forall h(1 \cdots H),$$

$$\frac{dV_h{}^*}{dt}(t) = -\alpha_h(\pi(t)) \cdot X_h(\pi(t)) \cdot \frac{{}^t d\pi}{dt}.$$

$\alpha_h(\Pi(t))$ is bounded from below, uniformly in t by $\bar{\alpha}_h$,[11]

$$\Rightarrow \frac{dV_h{}^*}{dt}(t) \geqslant \bar{\alpha}_h f(\pi(t), p(t)),$$

$$V_h{}^* = V_h{}^*(0) + \int_0^{+\infty} \frac{dV_h{}^*}{dt}(t)\, dt.$$

It is easy to see that there exists a sequence u_n such that:

u_n is a subsequence of $1 \cdots n$,
$u_1 = n_0$,
$\forall n: t_{u_n} + \mu \leqslant t_{u_{n+1}} - \mu$.

Hence:

$$V_h{}^* \geqslant V^*{}_h(0) + \int_0^{t_{n_0}-\mu} \frac{dV^*{}_h}{dt}(t)\, dt + \sum_n \int_{t_{u_n}-\mu}^{t_{u_n}+\mu} \frac{dV^*{}_h}{dt}(t)\, dt$$

$$\geqslant V^*{}_h(t_{n_0}-\mu) + \sum_n \int_{t_{u_n}-\mu}^{t_{u_n}+\mu} \bar{\alpha}_h f(\pi(t), p(t))\, dt$$

$$\geqslant V_h{}^*(t_{n_0}-\mu) + \sum_n \bar{\alpha}_h \frac{\epsilon}{4} 2\mu.$$

[11] $\alpha_h(\pi) = (\partial u_h/\partial x_k)/\pi_k$, $k(1 \cdots n) \Rightarrow \alpha_h(\pi) = \{\sum_{k=1}^n [(\partial u_h/\partial x_k)(X_h(\pi))]^2\}^{1/2}/\|\pi\|^2$, which is bounded on K by $\alpha_h > 0$ ($H2$).

This is impossible, which proves Proposition 2. A straightforward modification of the above argument proves Theorem 2.

Proof of Proposition 3

· From Proposition 1 a limit point $(\bar{\pi}, \bar{p})$ of system (1′) satisfies $\eta(\bar{p}) = X(\bar{\pi})$.

· Furthermore a limit point is such that the linear system

$$X_h(\bar{\pi}) \cdot {}^t a < 0 \qquad \forall h \in [1 \cdots H],$$
$$\bar{p} \cdot \partial\bar{X}(\bar{\pi}) \cdot {}^t a = 0$$

is inconsistent. It is equivalent to say that the system of inequalities

$$X_h(\bar{\pi}) \cdot {}^t a < 0,$$
$$\bar{p} \cdot \partial\bar{X}(\bar{\pi}) \cdot {}^t a \leqslant 0,$$
$$-\bar{p} \cdot \partial\bar{X}(\bar{\pi}) \cdot {}^t a \leqslant 0$$

is inconsistent.

From Rockafellar [12, Theorem 22.2], there exists:

$$\mu_h \geqslant 0, \qquad 1 \leqslant h \leqslant H,$$
$$\nu \geqslant 0,$$
$$\nu' \geqslant 0,$$

such that

at least one of the numbers $\mu_1 \cdots \mu_H$ is nonzero, and
$\sum_h \mu_h X_h(\bar{\pi}) + (\nu - \nu')\, \bar{p} \cdot \partial\bar{X}(\bar{\pi}) = 0$.

It is impossible that $\nu - \nu' = 0$ because of H4 and the assumption of specialized commodities H1.

Hence either $\nu - \nu' > 0$ and $(\bar{\pi}, \bar{p})$ is a local Pareto equilibrium, or $\nu - \nu' < 0$ and $\bar{p} \cdot \partial\bar{X}(\bar{\pi}) \cdot {}^t a \leqslant 0$ is then a consequence of $X_h(\bar{\pi}) \cdot {}^t a < 0$ $\forall h \in [1 \cdots H]$.

In other words $\mathring{K}(\bar{\pi}) \cap Q(\bar{\pi}, \bar{p}) \neq \varnothing$.

3

Proof of Proposition 4

First we verify that

$$U(\pi, p, \lambda) = \partial\bar{X}(\Pi) \cdot \frac{{}^t d\pi}{dt} + \frac{d\lambda}{dt}\, {}^t\pi + \lambda \frac{{}^t d\pi}{dt}$$

belongs to $V(p)$ so that dp/dt is well defined:

$$p \cdot U(\pi, p, \lambda) = p \cdot \partial\bar{X}(\pi) \cdot \frac{{}^t d\pi}{dt} - p \cdot {}^t\pi \frac{p \cdot \partial\bar{X}(\pi) + \lambda p}{p \cdot {}^t\pi} \cdot \frac{{}^t d\pi}{dt} + \lambda p \cdot \frac{{}^t d\pi}{dt} = 0.$$

Thus we have:

$$\partial\bar{\bar{\eta}}(p(t)) \cdot \frac{{}^t dp}{dt} = \partial\bar{X}(\pi(t)) \cdot \frac{{}^t d\pi}{dt} + \frac{d}{dt}(\lambda(t)\, {}^t\pi(t))$$
$$\Rightarrow \eta(p(t)) = X(\pi(t)) + \lambda(t)\, {}^t\pi(t) + C$$

where C is a constant which is in fact zero because of the initial conditions

$$\Rightarrow \forall t \in [0, T] : \eta(p(t)) = X(\pi(t)) + \lambda(t)\, {}^t\pi(t),$$

so we have proved the second assertion.

The third one is simply a consequence of : $d\Pi/dt \in \mathring{K}(\Pi(t))$.

In order to prove the first one, let us suppose that $\exists t_1 : \lambda(t_1) < 0$. Since $\lambda(0) \geqslant 0$, there exists $t_0 \geqslant 0$ such that $\lambda(t_0) = 0$ and for $t \in [t_0, t_1]$, $\lambda(t) \leqslant 0$.

From the definition of $Q(\Pi, p, \lambda)$ we have:

$$-\frac{p \cdot \partial\bar{X}(\pi) \cdot {}^t d\pi/dt}{p \cdot {}^t\Pi} \geqslant -\frac{\lambda \| p \|}{p \cdot {}^t\pi} \Leftrightarrow \frac{d\lambda}{dt} \geqslant \frac{-\lambda(\| p \| + p \cdot {}^t d\pi/dt)}{p \cdot {}^t\Pi}.$$

On $[t_0, t_1]$ we have $-\lambda \geqslant 0$. We always have $p \cdot {}^t\pi > 0$.

The Cauchy–Schwartz inequality proves that:

$$\| p \| + p \cdot \frac{{}^t d\pi}{dt} \geqslant 0.$$

Thus on$[t_0, t_1]$: $d\lambda/dt \geqslant 0$ which contradicts $\lambda(t_1) < 0$.

Proof of Theorem 3

The idea underlining the proof is as in Theorem 1, but one has to deal with the additional variable $\lambda \in [0, +\infty[$.

Step 1. System (4) can be written:

$$\left(\frac{d\pi}{dt}, \frac{{}^t dp}{dt}, \frac{d\lambda}{dt}\right) \in G(\pi, p, \lambda),$$

where:

$$G(\pi, p, \lambda) = \Big\{\Big(a,\ \partial\bar{\eta}^{-1}(p) \cdot \partial\bar{X}(\pi) \cdot {}^t a - \frac{p \cdot \partial\bar{X}(\pi) + \lambda p}{p \cdot {}^t\pi} \cdot {}^t a(\partial\bar{\eta}^{-1}(p) \cdot {}^t\pi) + \lambda\, \partial\bar{\eta}^{-1}(p) \cdot {}^t a,\ - \frac{p \cdot \partial\bar{X}(\pi) + \lambda p}{p \cdot {}^t\pi} \cdot {}^t a\Big),\quad a \in M(\pi, p, \lambda)\Big\}.$$

In order to apply Castaing's theorem we will prove that:

· For any open, relatively compact set $K \subset S$ such that $\bar{K} \subset S$, there exists ϵ such that φ is continuous on $K \times]-\epsilon, +\infty[$, and hence M is uhc on this set.

· Hence, any compact set in $S \times [0, +\infty[$ is contained in an open set where G is uhc and bounded.

(1) $\forall K \subset S, \exists \epsilon > 0$: $\forall(\pi, p, \lambda) \in K \times]-\epsilon, +\infty[$, $\varphi(\pi, p, \lambda) \neq \varnothing$. It is obvious for $\lambda \geqslant 0$. When $\lambda < 0$:

$$\forall a \in Q(\pi, p, \lambda) \qquad \| a \| \geqslant \frac{|\lambda| \| p \|}{\| p \cdot \partial\bar{X}(\pi) \|}$$
$$\Rightarrow \varphi(\pi, p, \lambda) \neq \varnothing \Leftrightarrow \frac{|\lambda| \| p \|}{\| p \cdot \partial\bar{X}(\pi) \|} \leqslant 1$$
$$\Leftrightarrow |\lambda| \leqslant \frac{\| p \cdot \partial\bar{X}(\pi) \|}{\| p \|}$$

but on $\bar{K}$ the continuous function $(\pi, p) \to \| p \cdot \partial\bar{X}(\pi) \| / \| p \|$ has a minimum $\epsilon > 0$.

And hence,

$$\varphi(\pi, p, \lambda) \neq \varnothing \qquad \forall(\pi, p, \lambda) \in K \times]-\epsilon, +\infty[.$$

Note now that φ is continuous on $K \times]-\epsilon, +\infty[$: it is upper-hemi-continuous as the intersection of upper-hemi-continuous correspondences and lower-hemi-continuous (see Proposition 6 in Appendix 2).

The maximum theorem then implies that M is upper-hemi-continuous and compact valued on $K \times]-\epsilon, +\infty[$, and hence G is uhc, compact valued, and clearly convex valued.

(2) For any compact set C in $S \times [0, +\infty[$, there exists an open, relatively compact neighborhood $K \times]-\epsilon, c[$ of C such that $\bar{K} \subset S$ and G is upper-hemi-continuous and bounded on $K \times]-\epsilon, c[$.

Hence by Castaing's theorem, there exists $T_c > 0$ such that for all $(\pi^0, p^0, \lambda^0) \in C$ there is a solution $(\pi(t), p(t), \lambda(t))$ of system (4) defined on $[0, T_c]$ and starting from (π^0, p^0, λ^0).

Step 2. Let $(\pi^0, p^0, \lambda^0) \in S \times [0, +\infty[$.

As in Section 2 we consider a sequence of nondecreasing compact sets C^k such that $S \times [0, +\infty[= \bigcup_k C^k$.

We then build sequences $C^{k(n)}$, $T_{k(n)}$, $\pi_n(t)$, $p_n(t)$, $\lambda_n(t)$ such that: $C^{k(n)}$ is the smallest compact set containing $(\pi_{n-1}(T_{k(n-1)})$, $p_{n-1}(T_{k(n-1)})$, $\lambda_{n-1}(T_{k(n-1)}))$, $(\pi_n(t), p_n(t), \lambda_n(t))$ is a solution path of system (4) defined on $[0, T_{k(n)}]$ and starting from $\pi_{n-1}(T_{k(n-1)})$, $p_{n-1}(T_{k(n-1)})$, $\lambda_{n-1}(T_{k(n-1)})$.

Now, we have to prove that the sequence $\lambda_n(T_{k(n)})$ is bounded.
According to Proposition 4. we have:

$$\forall t \in [0, T_{k(n)}] : \eta(p_n(t)) = X(\pi_n(t)) + \lambda_n(t)\, {}^t\pi_n(t),$$
$$\lambda_n(t) \geqslant 0$$

and

$$\|\pi_n(t)\| = \|\pi^0\|,$$
$$\|p_n(t)\| = \|p^0\|$$

$$\Rightarrow \|\lambda_n(t)\, {}^t\pi_n(t)\| = \|\eta(p_n(t)) - X(\pi_n(t))\|$$

$$\Rightarrow \lambda_n(t) = \frac{\|\eta(p_n(t)) - X(\pi_n(t))\|}{\|\pi_n(t)\|} = \frac{\|\eta(p_n(t)) - X(\pi_n(t))\|}{\|\pi^0\|}$$

but for all n, $(X(\pi_n(t)), \eta(p_n(t)))$ is a feasible state, hence is in a compact and thus bounded set.

This implies: $\lambda_n(t)$ is bounded (uniformly).

Hence one can extract a convergent subsequence from $\lambda_n(T_{k(n)})$ and the end of the proof goes on as in step 2 of Theorem 1.

When one has a trajectory for system (4) it is possible to build a trajectory for system (3) in the same way as in Section 2.

Proof of Theorem 4

The proof is a straightforward modification of the proof of Proposition 2 using the uniform continuity of $f(\pi, p, \lambda)$ on the compact set:

$$E(k_1, k_2, r_1, ..., r_H) = \{(\pi, p, \lambda) \in P \times P \times [0, +\infty[: \|\pi\| = k_1, \|p\| = k_2,$$
$$\eta(p) = X(\pi) + \lambda^t\pi,$$
$$u_h(X_h(\pi)) \geqslant r_h, \forall h\}$$

where all r_h are strictly positive.

Appendix 2

The theorem on which our proof of trajectories' existence relies is the following:

Castaing's Theorem [2]. *Let us consider the multivalued differential equation*

$$(E): \quad dx/dt \in F(t, x), \qquad t \in [0, a], \qquad x \in \Omega \text{ nonempty open set of } \mathbb{R}^n.$$

We suppose that:

(1) *$F(t, x)$ is a nonempty convex compact set of $\mathbb{R}^n$, $\forall t \in [0, a]$, $\forall x \in \Omega$.*

(2) *$\forall t \in [0, a]$, $x \to F(t, x)$ is upper-hemi-continuous on Ω.*

(3) *$\forall x \in \Omega$, $t \to F(t, x)$ is Lebesgue measurable on $[0, a]$.*

(4) *There exists a function g integrable on $[0, a]$ such that:*

$$\|u\| \leqslant g(t), \qquad \forall u \in F(t, x), \forall t \in [0, a], \forall x \in \Omega.$$

A solution of the differential equation (E) is a function X from $[0, t_0]$ (with $t_0 \leqslant a$) to Ω such that X is absolutely continuous and $dX(t)/dt \in F(t, X(t))$ a.e. on $[0, t_0]$.

Let M be any nonempty convex compact set in Ω, and $t_0 \in]0, a]$ such that $\int_0^{t_0} g(s)\, ds \leqslant d(M, C\Omega)$ (distance from M to the complementary set of Ω).

Theorem. *For any $\xi \in M$, there exists at least one solution X of the differential equation (E) on $[0, t_0]$ such that $X(0) = \xi$. The set S_ξ of all solutions X such that $X(0) = \xi$ is compact in the Banach space $C_{\mathbb{R}^n}[0, t_0]$.*[12]

Furthermore, the correspondence $\xi \to S_\xi$ is upper-hemi-continuous on M.

Here, we use only the first part of the theorem. Our function F does not depend on t so that we need not verify (3), and we can take for g any constant k such that:

$$\|u\| \leqslant k, \qquad \forall x \in \Omega.$$

Proposition 5. *Let φ and Ψ be (nonempty) correspondences defined by:*

$$\forall x \in \Omega \qquad \varphi(x) = \{y \in \mathbb{R}^n : a(x) \cdot y = b(x)\},$$
$$\Psi(x) = \{y \in \mathbb{R}^n : c(x) \cdot y = d(x)\}$$

where Ω is a set in $\mathbb{R}^p$,

a and c are continuous functions from Ω to $\mathbb{R}^n$,

b and d are continuous functions from Ω to $\mathbb{R}$,

$a(x)$ and $c(x)$ are independent vectors of $\mathbb{R}^n$, for all $x \in \Omega$;

then $\varphi \cap \Psi$ is a lower-hemi-continuous (lhc) correspondence on Ω.

Proof of Proposition 5. One has to show that:

For any sequence x^k in Ω converging to $\bar{x} \in \Omega$, for any $\bar{y} \in \varphi(\bar{x}) \cap \Psi(\bar{x})$, there exists a sequence $y^k \in \varphi(x^k) \cap \Psi(x^k)$ converging to $\bar{y}$.

[12] Where $C_{\mathbb{R}^n}[0, t_0]$ is the set of all continuous functions from $[0, t_0]$ to $\mathbb{R}^n$, endowed with the topology of uniform convergence.

$\bar{y}$ is a solution of the system:

$$\begin{aligned} a_1(\bar{x}) \cdot y_1 + a_2(\bar{x}) \cdot y_2 + \cdots + a_n(\bar{x}) \cdot y_n &= b(\bar{x}); \\ c_1(\bar{x}) \cdot y_1 + c_2(\bar{x}) \cdot y_2 + \cdots + c_n(\bar{x}) \cdot y_n &= d(\bar{x}). \end{aligned} \tag{S}$$

As $a(\bar{x})$ and $c(\bar{x})$ are independent there exists a matrix of order 2 in

$$\begin{pmatrix} a_1(\bar{x}) \cdots a_n(\bar{x}) \\ c_1(\bar{x}) \cdots c_n(\bar{x}) \end{pmatrix}$$

which is of full rank 2.

For example, let us suppose that:

$$D(\bar{x}) = \begin{vmatrix} a_1(\bar{x}) & a_2(\bar{x}) \\ c_1(\bar{x}) & c_2(\bar{x}) \end{vmatrix} \neq 0.$$

System (S) can then be written:

$$a_1(\bar{x}) \cdot y_1 + a_2(\bar{x}) \cdot y_2 = b(\bar{x}) - a_3(\bar{x}) \cdot y_3 - \cdots - a_n(\bar{x}) \cdot y_n ;$$
$$c_1(\bar{x}) \cdot y_1 + c_2(\bar{x}) \cdot y_2 = d(\bar{x}) - c_3(\bar{x}) \cdot y_3 - \cdots - c_n(\bar{x}) \cdot y_n .$$

Let us denote

$$\tilde{b}(x, y_3 \cdots y_n) = b(x) - a_3(x) \cdot y_3 - \cdots - a_n(x) \cdot y_n ;$$
$$\tilde{d}(x, y_3 \cdots y_n) = d(x) - c_3(x) \cdot y_3 - \cdots - c_n(x) \cdot y_n .$$

$\tilde{b}$ and $\tilde{d}$ are clearly continuous.

And one knows that $\bar{y}$, as a solution of system (S), can be written:

$$\bar{y}_1 = \frac{\begin{vmatrix} \tilde{b}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) & a_2(\bar{x}) \\ \tilde{d}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) & c_2(\bar{x}) \end{vmatrix}}{D(\bar{x})},$$

$$\bar{y}_2 = \frac{\begin{vmatrix} a_1(\bar{x}) & \tilde{b}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) \\ c_1(\bar{x}) & \tilde{d}(\bar{x}, \bar{y}_3, \ldots, \bar{y}_n) \end{vmatrix}}{D(\bar{x})}.$$

As $D(\bar{x}) \neq 0$ and D is a continuous function, there exists a neighborhood $\mathscr{V}$ of $\bar{x}$ such that:

$$\forall x^k \in \mathscr{V} \qquad D(x^k) \neq 0.$$

Then let $y^k = (y_1^k, y_2^k, \bar{y}_3, \ldots, \bar{y}_n)$ be defined by:

$$y_1^k = \frac{\begin{vmatrix} \tilde{b}(x^k, \bar{y}_3, \ldots, \bar{y}_n) & a_2(x^k) \\ \tilde{d}(x^k, \bar{y}_3, \ldots, \bar{y}_n) & c_2(x^k) \end{vmatrix}}{D(x^k)};$$

$$y_2^k = \frac{\begin{vmatrix} a_1(x^k) & \tilde{b}(x^k, \bar{y}_3, \ldots, \bar{y}_n) \\ c_1(x^k) & \tilde{d}(x^k, \bar{y}_3, \ldots, \bar{y}_n) \end{vmatrix}}{D(x^k)}.$$

Then y^k is a solution of

$$a(x^k) \cdot y = b(x^k),$$
$$c(x^k) \cdot y = d(x^k),$$

i.e., $y^k \in \varphi(x^k) \cap \Psi(x^k)$ and $y^k \to \bar{y}$, by the continuity of all functions $\tilde{b}$, $\tilde{d}$, a_1, a_2, c_1, c_2, and D.

COROLLARY 1. *The correspondence* φ *defined on* S *by*:

$\varphi(\Pi, p) = \{a \in \mathbb{R}^n / p \cdot \partial\overline{X}(\Pi) \cdot {}^t a = 0, \Pi \cdot {}^t a = 0, \| a \| \leqslant 1\}$ *is* lhc *on* S.

To prove that, take:

$$a(\Pi, p) = p \cdot \partial\overline{X}(\Pi),$$
$$b(\Pi, p) = d(\Pi, p) = 0,$$
$$c(\Pi, p) = \Pi,$$

and $\Omega = S$.

a and c are independent because they are normal (homogeneity of X) and nonzero on S.

Using Proposition 5 we have:

$$\forall(\pi_k, p_k) \in S: (\pi_k, p_k) \xrightarrow[k \nearrow +\infty]{} (\bar{\pi}, \bar{p}) \in S.$$

$$\forall \bar{a} \in \varphi(\bar{\pi}, \bar{p}), \exists a_k \to \bar{a} : \pi_k \cdot \partial\overline{X}(\pi_k) \cdot {}^t a_k = 0,$$
$$\pi_k \cdot {}^t a_k = 0.$$

If $\| a_k \| \leqslant 1$, for all k, then $a_k \in \varphi(\pi_k, p_k)$ and the corollary follows.

If $\| a_k \| > 1$ for some k (which can only happen, in a significant way, when $\| \bar{a} \| = 1$).

Let $b_k = a_k / \| a_k \|$, $b_k \to \bar{a} / \| \bar{a} \| = \bar{a}$,

$$p_k \cdot \partial\overline{X}(\pi_k) \cdot {}^t b_k = 0, \qquad \text{as well,}$$

and

$$\pi_k \cdot {}^t b_k = 0,$$

hence $b_k \in \varphi(\pi_k, p_k)$ which proves the corollary.

PROPOSITION 6. *Let* φ *and* Ψ *be (nonempty) correspondences defined by*:

$$\forall x \in \Omega : \varphi(x) = \{ y \in \mathbb{R}^n : a(x) \cdot y = b(x)\},$$
$$\Psi(x) = \{ y \in \mathbb{R}^n : c(x) \cdot y \leqslant d(x)\},$$

where Ω, a, b, c, and d are as in Proposition 5. Then $\varphi \cap \Psi$ is a lower-hemi-continuous correspondence.

Proof of Proposition 6. Let x^k be a sequence in Ω converging to some $\bar{x} \in \Omega$ and $\bar{y} \in \varphi(\bar{x}) \cap \Psi(\bar{x})$.

· Either $\bar{y}$ belongs to the interior of $\Psi(\bar{x})$, then, as φ is lower-hemi-continuous, there exists a sequence $y^k \in \varphi(x^k)$ converging to $\bar{y}$ and for k large enough y^k also belongs to $\Psi(x^k)$.

· Or $\bar{y}$ belongs to the frontier of $\Psi(\bar{x})$, and we know from Proposition 5 that we can build a sequence y^k converging to $\bar{y}$ such that

$$\forall k : a(x^k) \cdot y^k = b(x^k),$$
$$c(x^k) \cdot y^k = d(x^k).$$

COROLLARY 2. *The correspondence* φ *defined on* $S \times]-\epsilon, +\infty[$ *by* $\varphi(\Pi, p, \lambda) = \{a \in \mathbb{R}^n : p \cdot \partial\bar{X}(\Pi) \cdot {}^t a \leqslant \lambda \| p \|, \Pi \cdot {}^t a = 0, \| a \| \leqslant 1\}$ *is* lhc *on* $S \times]-\epsilon, +\infty[$.

Let us take:

$$a(\Pi, p, \lambda) = p \cdot \partial\bar{X}(\Pi),$$
$$b(\Pi, p, \lambda) = \lambda \| p \|,$$
$$c(\Pi, p, \lambda) = \Pi,$$
$$d(\Pi, p, \lambda) = 0.$$

According to Proposition 6:

$$\forall(\Pi_k, p_k) \in S: \quad (\Pi_k, p_k) \xrightarrow[k \nearrow +\infty]{} (\bar{\Pi}, \bar{p}) \in S,$$
$$\forall \lambda_k \in]-\epsilon, +\infty[: \quad \lambda_k \xrightarrow[k \nearrow +\infty]{} \bar{\lambda} \in]-\epsilon, +\infty[,$$
$$\forall \bar{a} \in \varphi(\bar{\Pi}, \bar{p}, \bar{\lambda}), \exists a_k \to \bar{a}: \quad p_k \cdot \partial\bar{X}(\Pi_k) \cdot {}^t a_k \leqslant \lambda_k \| p_k \|, \quad \Pi_k \cdot {}^t a_k = 0.$$

The problem is to find a sequence in the unit ball $B(0, 1)$. If $\| \bar{a} \| < 1$, then from a rank k on, $\| a_k \| < 1$, which solves our problem.

Now, if $\| \bar{a} \| = 1$, there are three possibilities:

(1) $\bar{\lambda} > 0 \Rightarrow \exists k_0 \in \mathbb{N} : \forall k \geqslant k_0, \lambda_k > 0$.

Take

$$\mu_k = \begin{cases} 1 & \forall k: \| a_k \| \leqslant 1, \\ 1/\| a_k \| & \forall k: \| a_k \| > 1, \end{cases}$$

and $b_k = \mu_k a_k$.

It is clear that:

$$b_k \xrightarrow[k \nearrow +\infty]{} \bar{a},$$
$$\| b_k \| \leqslant 1,$$
$$p_k \cdot \partial\bar{X}(\Pi_k) \cdot {}^t b_k \leqslant \lambda_k \| p_k \|,$$
$$\Pi_k \cdot {}^t b_k = 0,$$

hence $b_k \in \varphi(\Pi_k, p_k, \lambda_k)$ which ends the proof for this case.

(2) $\bar{\lambda} < 0 \Rightarrow \exists k_0 \in \mathbb{N}, \forall k \geqslant k_0, \lambda_k < 0$.

It is possible to build a sequence c_k of the form $c_k = \mu_k p_k \cdot \partial\bar{X}(\Pi_k) + \rho_k e_k$ where e_k lies in the hyperplane normal to $p_k \cdot \partial\bar{X}(\Pi_k)$, and such that:

$$\forall k, \quad c_k \in \varphi(\Pi_k, p_k, \lambda_k) \quad \text{and} \quad c_k \xrightarrow[k \nearrow +\infty]{} \bar{a}.$$

The complete proof is left to the reader.

(3) $\bar{\lambda} = 0$.

For those k where $\lambda_k > 0$ we take $f_k = b_k$ and for those k where $\lambda_k < 0$ we take $f_k = c_k$. Then it is clear that:

$$\forall k, f_k \in \varphi(\Pi_k, p_k, \lambda_k) \quad \text{and} \quad f_k \to_{k \nearrow +\infty} \bar{a},$$

which ends the proof of Corollary 2.

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