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ASYMPTOTIC VALUE OF MIXED GAMES*

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In this paper we are concerned with *mixed games*, i.e., games with on one hand an "ocean" of insignificant players (formalized by a continuum of players) and on the other hand some significant players (atoms). Considering these games as limits of finite games, we show, for the subset *pFL*, that the Shapley-Hart value of the mixed game corresponding to the uniform probability measure is the limit of the Shapley values of the associated finite games. This paper should then be considered as a generalization of the results of the work by Aumann-Shapley on nonatomic games.

I. Introduction. In 1960, Shapiro and Shapley proved a limit theorem on *weighted majority games* (1978), namely that if the players are separated into two categories: N, the major players on the one hand (who are supposed to be in finite number), and M, the minor players on the other hand, and if the number of minor players increases (their weights decreasing in an appropriate fashion) then the value of a major player in the (finite) game goes to a fixed limit (which does not depend on the way the weights of minor players go to zero).

In 1961 Milnor and Shapley (1978) introduced for majority games a notion of infinite game: the so-called "oceanic" games where the players are partitioned in "major players" (in finite number) and an "ocean" of infinitesimal minor players. They then used the majority rule to define a "value" for this game by generalizing the Shapley value of finite majority games.

The notion of value for infinite games was next precisely formalized and studied in Aumann-Shapley's book (1974). But, there, they are mainly concerned with non-atomic games, i.e., games where all players are infinitesimal.

From these contributions (see also a paper by Kannai (1966)) it appears clear that the notion of value for infinite games can be tackled in two different ways:

—the axiomatic approach which would define a value by a set of axioms.

—the asymptotic approach which consists in considering the infinite game as the limit of finite games and then calculating its value as the limit of their Shapley values.

It has been proved by Aumann-Shapley (1974) that on a particular set of nonatomic games these two approaches coincide.

Now, for mixed games, i.e., games where the set of players is made of major players (atoms) on the one hand and minor players (ocean) on the other, the axiomatic study was made by Hart (1973). He proved that for these games it is no longer possible to speak of *the* value of a game: there is an infinity of values, one for each probabilistic way of "mixing" the atoms into the infinitesimal players.

However, the Milnor-Shapley study (1978) in the case of majority games suggested that the asymptotic approach would lead to selecting the value associated with the uniform probability measure.

In this paper, we use the formalism of Hart (1973) and Aumann-Shapley (1974) to make the asymptotic study and we prove that on a particular subset of mixed games

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the two approaches coincide, in the case where the value associated to the uniform probability measure is used.

In §2, some definitions and notation are given. In §3, the main theorem is stated and proved.

II. Preliminaries. All definitions and notation are as in Aumann-Shapley (1974) and Hart (1973).

Let $(I, \mathcal{C})$ be a measurable space assumed to be isomorphic to $([0, 1], \mathcal{B})$ where $\mathcal{B}$ is the σ-field of Borel sets on $[0, 1]$: elements of I are *players*, those of $\mathcal{C}$ are *coalitions*.

A set function or game is a real valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$.

A game v is monotonic if: $\forall S, T \in \mathcal{C},\ S \subset T \Rightarrow v(S) \leqslant v(T)$ (if Q is a space of games, Q^+ will denote the space of monotonic games in Q).

A game v is *of bounded variation* if it is the difference between two monotonic games. The space of all games of bounded variation is called BV.

On BV we define a norm, which we call the *variation norm* by: $\|v\| = \inf\{u(I) + w(I)/u, w \in BV^+, v = u - w\}$, for all v in BV. In the following we will always be concerned with the topology induced by the variation norm.

The space of all real valued functions f of bounded variation on $[0, 1]$ such that: $f(0) = 0$ is called bv. bv' will denote the space of those functions in bv which are continuous at 0 and 1.

A *carrier* of a game v is a coalition N such that: $\forall S \in \mathcal{C},\ v(S) = v(S \cap N)$.

The subspace of BV consisting of all measures with a finite carrier will be denoted FC.

A coalition S is *null* if its complement is a carrier and a player s is null if $\{s\}$ is null. A game is *nonatomic* if any player is null.

We shall denote by NA the subspace of BV consisting of all nonatomic measures and $FL = NA + FC$ (i.e., FL is the subspace of all measures that can be written as the sum of two measures, one nonatomic and the other with a finite carrier).

We call:

- $bv'NA$ (resp. $bv'FL$) the closed subspace of BV spanned[1] by games of the form $f \circ \omega$, where $f \in bv'$ and $\omega \in NA^+$ (resp. $\omega \in FL^+$) is a probability measure. Games in $bv'FL$ are called *mixed games*.

A mixed game is thus any game of the form $f \circ \omega$—or linear combination or limit of such games—where $\omega = \mu + \nu$, $\mu \in NA^+$, $\nu \in FC^+$. The finite carrier of ν will represent the major or significant players (or atoms). The measure μ will represent the "weights" of the minor players or "ocean."

- pNA (resp. pFL) the closed subspace of $bv'NA$ (resp. $bv'FL$) spanned by all powers of measures in NA^+ (resp. FL^+).

We shall need the following result of Hart (1973):

THEOREM A. *Let p be a continuous probability measure*[2] *on* $([0, 1], \mathcal{B})$. *Then there exists a value Φ on $bv'FL$ such that*: $\forall f \in bv', \forall \omega \in FL^+, \Phi v = \Phi_p v$ *where* $v = f \circ \omega$ (*i.e., there exists a value Φ on $bv'FL$ whose restriction to games of the form $f \circ \omega$ is Φ_p*) *where Φ_p is defined in the following way*:

let $\mu \in NA^+$ and $\nu \in FC^+$ be such that: $\omega = \mu + \nu$ *and denote by N the* (*finite*) *carrier of ν* (*n its cardinality*);

let Π be the set of all one-to-one mappings from N onto $J_n = \{1, \ldots, n\}$:

let $T_1, \ldots, T_n$ be n independent, equally distributed random variables, with distribu-

[1] The space spanned by $A \subset BV$ is the closure (in the variation norm topology) of the vector space generated by A.

[2] Its distribution function is continuous.

tion p, and $T^{(1)}, \ldots, T^{(n)}$ $(0 < T^{(1)} \leqslant \ldots \leqslant T^{(n)} < 1)$ *the corresponding order statistics.*

Then the set function $\Phi_p v$ *is defined by*:

$$\begin{cases} \Phi_p v(\{s\}) = E\left[\frac{1}{n!} \sum_{\pi \in \Pi} \Delta(\pi s, T^{(\pi s)}, \nu \circ \pi^{-1}, f)\right] & \forall s \in N, \\ \Phi_p v(S) = \alpha\mu(S) & \forall S \subset I/N, \end{cases}$$

with

$$\alpha = \begin{cases} \dfrac{f(1) - \Phi_p v(N)}{\mu(I)} & \textit{if } \mu(I) > 0, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\Delta(i, t, \xi, f) = f[t(1 - \xi(J_n)) + \xi(J_i)] - f[t(1 - \xi(J_n)) + \xi(J_{i-1})]$$
$$(\textit{where } J_k = \{1, \ldots, k\}).$$

We will now introduce the notion of *asymptotic value* (see Aumann-Shapley (1974) —chapter III). A *partition* π of I is a finite family of disjoint subsets whose union is I. A partition π_2 is a *refinement* of a partition π_1 if each member of π_1 is a union of members of π_2. A partition is *measurable* if each of its members is measurable. A sequence $(\Pi_1, \Pi_2 \ldots)$ of partitions is *decreasing* if Π_{i+1} is a refinement of Π_i for all i. This sequence is *separating* if $\forall s, t \in I, s \neq t, \exists i : s$ and t are in different members of Π_i. A decreasing separating sequence of measurable partitions will be called an *admissible sequence*. Let v be a game and Π a measurable partition of I. We denote v_Π the finite game whose players are the members $A_1, \ldots, A_m$ of Π, given by:

$$\forall \Xi \subset \Pi : v_\Pi(\Xi) = v\left(\bigcup_{A_j \in \Xi} A_j\right).$$

If u is a finite game, φu will denote its Shapley value (1953).

Let T be a measurable subset in I and $\mathcal{P} = (\Pi_1, \Pi_2 \ldots)$ a decreasing sequence of measurable partitions such that:

$$\Pi_1 = (T, I \setminus T),$$
$$\forall m : T_m = \{\Xi \in \Pi_m : \Xi \subset T\}.$$

T_m is a coalition in the finite game v_{Π_m} and thus has a value $\varphi v_{\Pi_m}(T_m)$; if this value has a limit when m goes to infinity one will denote:

$$\varphi_{\mathcal{P}} v(T) = \lim_{m \to +\infty} \varphi v_{\Pi_m}(T_m).$$

If $\varphi_{\mathcal{P}}(v(T))$ exists for all admissible $\mathcal{P}$ starting from Π_1 and is *independent* of $\mathcal{P}$ then we call it the *asymptotic value* $\varphi v(T)$.

If all $T \in \mathcal{B}$ possesses an asymptotic value then v is said to have an asymptotic value. We will denote: ASYMP the set of all $v \in BV$ having an asymptotic value.

III. The main theorem.

THEOREM 1. *pFL* $\subset$ ASYMP *and the value on pFL associated with the uniform probability on* [0, 1] *coincides with the asymptotic value.*

PROOF. *pFL* is spanned by *the* games of the form $f \circ \omega$ where f is a polynomial function in bv' and ω belongs to FL^+. The following two lemmas proved in Aumann-Shapley (1974) enable us to limit ourselves to study games of this form only.

LEMMA 1. $\|\varphi v\| \leqslant \|v\| \ \forall v \in \text{ASYMP}$.

LEMMA 2. ASYMP *is a closed linear subspace of* BV.

Hence, for proving our theorem, it is sufficient to show:
$\forall f$ polynomial function in bv'
$\forall \omega$ probability measure in FL^+:
(i) $f \circ \omega \in \text{ASYMP}$,
(ii) $\varphi(f \circ \omega) = \Phi(f \circ \omega)$,
where Φ denotes the value Φ_p associated with the uniform probability measure on [0, 1].

Let us quote now a classical property of order statistics that allows us to give the precise expression of Φ.

LEMMA 3.[3] *Let T_0 be a random variable whose distribution is uniform on* [0, 1]. *Let $(T_1, \dots, T_n)$ be a random sample of size n from T_0 and $(T^{(1)}, \dots, T^{(n)})$ be the associated order statistics. Then*:

- *The joint density of $(T^{(1)}, \dots, T^{(n)})$ is given by*:

$$g(t_1, \dots, t_n) = \begin{cases} n! & \text{if } 0 \leqslant t_1 \leqslant \dots \leqslant t_n \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

- *The marginal density of $T^{(i)}$ is given by*:

$$g_i(t) = \begin{cases} n! \dfrac{t^{i-1}(1-t)^{n-i}}{(i-1)!(n-i)!} & \text{on } [0, 1]. \\ 0 & \text{otherwise.} \end{cases}$$

Applying this lemma to

$$v = f \circ \omega,$$

$$\omega = \mu + \nu, \qquad \mu \in NA^+, \nu \in FL^+$$

we have:

$$\Phi v(\{s\}) = \sum_{\pi \in \Pi} \int_0^1 \Delta(\pi s, t, \nu \circ \pi^{-1}, f) \frac{t^{\pi s - 1}(1-t)^{n - \pi s}}{(\pi s - 1)!(n - \pi s)!} \, dt \quad \forall s \in N, \tag{1}$$

$$\Phi v(S) = \frac{f(1) - \Phi v(N)}{\mu(I)} \mu(S) \quad \forall S \subset I/N, \tag{2}$$

where N is the smallest carrier of ν.

Now let $A \subset I$ be a measurable subset of I. Separating the atoms in A we have:

$$A = S \cup \Bigg(\bigcup_{s \in A \cap N} \{s\} \Bigg) \quad \text{where } S \subset I/N.$$

Let us show that, for all admissible sequences $\mathcal{P} = (\Pi_1, \Pi_2, \dots)$ starting from $\Pi_1 = (A, I/A)$:

$$\varphi v_{\Pi_i}(A_i) \underset{i \uparrow +\infty}{\to} \Phi v(A) \tag{3}$$

where $A_i = \{j \in \Pi_i : j \subset A\}$ and $\Phi v(A) = \Phi v(S) + \sum_{s \in A \cap N} \Phi v(\{s\})$.

From the additivity of the value it will be sufficient to prove (3) first for a set S lying in I/N and then for an atom $s \in N$. So, let us take $S \subset I/N$ measurable and

[3] Cf. theorem VI, 11 and 12 in Mood-Graybill-Boes (1974).

prove that for all admissible sequences $\mathcal{P} = (\Pi_1, \Pi_2, \ldots)$ starting from $\Pi_1 = (S, I/S)$ we have:

$$\varphi v_{\Pi_i}(S_i) \underset{i \uparrow +\infty}{\rightarrow} \Phi v(S) \tag{4}$$

with $S_i = \{j \in \Pi_i : j \subset S\}$.

We shall suppose, without loss of generality that, for i sufficiently large, the atoms (i.e., the elements in N) are all separated in the partition Π_i, and hence that: $p_i = r_i + n$ where r_i denotes the number of members of Π_i in I/N and p_i the number of members of Π_i.

$$\varphi v_{\Pi_i}(S_i) = \sum_{j \in S_i} \varphi v_{\Pi_i}(\{j\}),$$

$$\varphi v_{\Pi_i}(\{j\}) = \sum_{T \supset j} \frac{(p_i - t)!(t-1)!}{p_i!} \left[v_{\Pi_i}(T) - v_{\Pi_i}(T - \{j\}) \right],$$

where T denotes a coalition of t players of the game v_{Π_i}.

We shall write: $T = R \cup L$ where R consists of all players of v_{Π_i} in I/N and L contains atoms, with $|R| = r$ and $|L| = l$.

$$\varphi v_{\Pi_i}(\{j\}) = \sum_{T \supset j} \frac{(p_i - t)!(t-1)!}{p_i!} \{ f[\mu(R) + \nu(L)] - f[\mu(R) + \nu(L) - \mu(j)] \}$$

$$= \sum_{L \subset N} \sum_{R \supset j} \frac{(p_i - r - l)!(r + l - 1)!}{p_i!} \mu(j) f'[\mu(R) + \nu(L) - \theta\mu(j)], \qquad 0 < \theta < 1,$$

$$= \mu(j) \sum_{L \subset N} \sum_{R \supset j} \left\{ \frac{(r_i - r)!(r-1)!}{r_i!} g_L'[\mu(R) - \theta\mu(j)] \right\} \times \frac{r_i!(p_i - r - l)!(r + l - 1)!}{p_i!(r_i - r)!(r-1)!}.$$

with

$$g_L(x) = f(x + \nu(L)).$$

Let

$$K = \frac{r_i!(p_i - r - l)!(r + l - 1)!}{p_i!(r_i - r)!(r-1)!}$$

$$= \frac{1}{\left(1 - \frac{1}{p_i}\right) \cdots \left(1 - \frac{n-1}{p_i}\right)} \sum_{\alpha=1}^{l} \sum_{\beta=0}^{n-l} \frac{1}{p_i^{n-\alpha-\beta}} C_\alpha C_\beta \left(1 - \frac{r}{p_i}\right)^\beta \left(\frac{r}{p_i}\right)^\alpha$$

where C_α and C_β only depend on l and n.

$$\Rightarrow \varphi v_{\Pi_i}(\{j\}) = \mu(j) \sum_{L \subset N} \sum_{\alpha=1}^{l} \sum_{\beta=0}^{n-l} \frac{C_\alpha C_\beta}{\left(1 - \frac{1}{p_i}\right) \cdots \left(1 - \frac{n-1}{p_i}\right) p_i^{n-\alpha-\beta}} \sum_{R \supset j} A_{\alpha\beta}(R)$$

with

$$A_{\alpha\beta}(R) = \frac{(r_i - r)!(r-1)!}{r_i!} g_L'[\mu(R) - \theta\mu(j)] \left(\frac{r}{p_i}\right)^\alpha \left(1 - \frac{r}{p_i}\right)^\beta.$$

We shall show that:

$$\sum_{R \supset j} A_{\alpha\beta}(R) - \frac{1}{r_i - 1} \frac{1}{C_{r_i-1}^{r-1}} g_L'\left[\frac{r-1}{r_i - 1}\mu(I)\right]\left(\frac{r-1}{r_i-1}\right)^{\alpha}\left(1 - \frac{r-1}{r_i-1}\right)^{\beta}$$

converges to 0 uniformly in j.

This is the same as showing that, given a partition $\mu(I) = \sum_{j=1}^{r_i - 1} \alpha_j$ and given a continuous function g on $[0, \mu(I)]$,

$$A = \sum_{R \supset j} \frac{(r_i - r)!(r-1)!}{r_i!}\left[g\left(\sum_{j \in R}\alpha_j\right)\left(\frac{r}{p_i}\right)^{\alpha}\left(1 - \frac{r}{p_i}\right)^{\beta}\right.$$

$$\left. - \frac{p_i - n}{p_i - n - 1} g\left(\frac{r-1}{r_{i-1}}\mu(I)\right)\left(\frac{r-1}{r_i-1}\right)^{\alpha}\left(1 - \frac{r-1}{r_i-1}\right)^{\beta}\right]$$

converges to 0 uniformly in j, and this follows easily from the proof in Kannai (1966).

Using the definition of the Riemann integral:

$$\int_0^1 g_L'[x\mu(I)]x^{\alpha}(1-x)^{\beta}dx$$

$$= \lim_{i \uparrow +\infty} \sum_r \frac{1}{r_i - 1} g_L'\left[\frac{r-1}{r_i-1}\mu(I)\right]\left(\frac{r-1}{r_i-1}\right)^{\alpha}\left(1 - \frac{r-1}{r_i-1}\right)^{\beta}$$

$$= \lim_{i \uparrow +\infty} \sum_{R \supset j} \frac{1}{(r_i - 1)C_{r_i-1}^{r-1}} g_L'\left[\frac{r-1}{r_i-1}\mu(I)\right]\left(\frac{r-1}{r_i-1}\right)^{\alpha}\left(1 - \frac{r-1}{r_i-1}\right)^{\beta}$$

$$\Rightarrow \sum_{R \supset j} A_{\alpha\beta}(R) \xrightarrow[i \uparrow +\infty]{} \int_0^1 g_L'[x\mu(I)]x^{\alpha}(1-x)^{\beta}dx \quad \text{uniformly in } j,$$

and hence,

$$\lim_i \varphi v_{\Pi_i}(\{j\}) = \sum_{L \subset N} \mu(j) \int_0^1 g_L'[x\mu(I)]x^{l}(1-x)^{n-l}dx.$$

(All the terms in $1/p_i^{n-\alpha-\beta}$ have limit 0 except for $n = \alpha + \beta$.)

$$\Rightarrow \lim_i \varphi v_{\Pi_i}(S_i) = \sum_{j \in S_i} \mu(j) \sum_{L \subset N} \int_0^1 f'[x\mu(I) + \nu(L)]x^{l}(1-x)^{n-l}dx$$

$$= \mu(T) \sum_{L \subset N} \int_0^1 f'[x\mu(I) + \nu(L)]x^{l}(1-x)^{n-l}dx$$

$$= \mu(T) \sum_{L \subset N}\left\{\left[x^{l}(1-x)^{n-l}\frac{f[x\mu(I) + \nu(L)]}{\mu(I)}\right]_0^1\right.$$

$$\left. - \int_0^1 \frac{x^{l-1}(1-x)^{n-l-1}(l - nx)}{\mu(I)} f[x\mu(I) + \nu(L)]dx\right\},$$

$$\lim \varphi v_{\Pi_i}(S_i) = \frac{\mu(T)}{\mu(I)} \left\{ f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \int_0^1 \frac{(1-x)^{n-\pi s-1} x^{\pi s - 1}(\pi s - nx)}{(\pi s)!(n-\pi s)!} \right.$$

$$\left. \times f\left[x\mu(I) + \nu \circ \pi^{-1}(J_{\pi s}) \right] dx \right\}$$

$$= \frac{\mu(T)}{\mu(I)} \left\{ f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \left\{ \int_0^1 \frac{(1-x)^{n-\pi s} x^{\pi s - 1}}{(\pi s - 1)!(n - \pi s)!} f\left[x\mu(I) + \nu \circ \pi^{-1}(J_{\pi s}) \right] dx \right.\right.$$

$$\left.\left. - \int_0^1 \frac{(1-x)^{n-\pi s-1} x^{\pi s}}{(\pi s)!(n-\pi s-1)!} f\left[x\mu(I) + \nu \circ \pi^{-1}(J_{\pi s - 1}) \right] dx \right\} \right\}.$$

$$\lim_i \varphi v_{\Pi_i}(S_i) = \frac{\mu(T)}{\mu(I)} \left[f(1) - \sum_{s \in N} \sum_{\pi \in \Pi} \int_0^1 \frac{(1-x)^{n-\pi s} x^{\pi s - 1}}{(\pi s - 1)!(n-\pi s)!} \Delta(\pi s, x, \nu \circ \pi^{-1}, f) dx \right]$$

$$= \frac{\mu(T)}{\mu(I)} \left[f(1) - \sum_{s \in N} \Phi v(\{s\}) \right]$$

$$= \Phi v(S),$$

and this proves the result (4) for all measurable $S \subset I/N$.

Now let $s \in N$ and $\Pi_1 = \{\{s\}, I/\{s\}\}$ be a partition of I. Let us show that, for all admissible sequences: $\mathcal{P} = (\Pi_1, \Pi_2, \ldots)$

$$\varphi v_{\Pi_i}(\{s\}) \xrightarrow[i \uparrow +\infty]{} \Phi v(\{s\}). \tag{5}$$

Using the same notation as before:

$$\text{card } \Pi_i = p_i = r_i + n,$$

we have:

$$\varphi v_{\Pi_i}(\{s\}) = \sum_{T \supset \{s\}} \frac{(p_i - t)!(t-1)!}{p_i!} \left[v_{\Pi_i}(T) - v_{\Pi_i}(T - \{s\}) \right].$$

Define $T = R \cup L$ where R contains all players of v_{Π_i} in I/N and L consists of the atoms:

$$\varphi v_{\Pi_i}(\{s\}) = \sum_{L \supset \{s\}} \sum_{R \subset I/N} \frac{(p_i - t)!(t-1)!}{p_i!} \left\{ f[\mu(R) + \nu(L)] \right.$$

$$\left. - f[\mu(R) + \nu(L) - \nu(\{s\})] \right\}$$

$$= \sum_{L \supset \{s\}} \sum_{R \subset I/N} \frac{(p_i - t)!(t-1)! r_i!}{p_i! r! (r_i - r)!} \left\{ \frac{r!(r_i - r)!}{r_i!} \left(g_L[\mu(R)] - h_L[\mu(R)] \right) \right\}$$

with $g_L(x) = f[x + \nu(L)]$, $h_L(x) = f[x + \nu(L) - \nu(\{s\})]$; and by a proof as in (3) we have:

$$\varphi v_{\Pi_i}(\{s\}) - \sum_{L \supset \{s\}} \sum_{r=0}^{r_i} \frac{1}{r_i} \left(1 - \frac{r}{r_i}\right)^{n-l-1} \left(\frac{r}{r_i}\right)^l \left\{ g_L\left[\frac{r}{r_i}\mu(I)\right] - h_L\left[\frac{r}{r_i}\mu(I)\right] \right\}$$

converges to 0.

Hence, recalling the definition of the Riemann integral:

$$\lim_i \varphi v_{\Pi_i}(\{s\}) = \sum_{L \supset \{s\}} \int_0^1 (1-x)^{n-l-1} x^l \{ g_L[x\mu(I)] - h_L[x\mu(I)] \} dx$$

$$= \sum_{L \supset \{s\}} \int_0^1 \frac{x^l (1-x)^{n-l-1}}{l!(n-l-1)!} l!(n-l-1)! \{ f[x\mu(I) + \nu(L)]$$

$$-f[x\mu(I) + \nu(L) - \nu(\{s\})]\} dx$$

$$= \sum_{\pi \in \Pi} \int_0^1 \frac{x^{\pi s - 1}(1-x)^{n-\pi s}}{(\pi s - 1)!(n - \pi s)!} \{ f[x\mu(I) + \nu \circ \pi^{-1} J_{\pi s}]$$

$$-f[x\mu(I) + \nu \circ \pi^{-1} J_{\pi s - 1}]\} dx$$

$$= \Phi v(\{s\}).$$

This proves (5) and ends the proof of (3) and thus of our theorem.

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