INCOMPLETE MARKETS

Volume II: Infinite Horizon Economies

Part I: Existence of Equilibrium, Ponzi Schemes, and Asset Price Bubbles

1. Magill, M. and M. Quinzii (1994), "Infinite Horizon Incomplete Markets", *Econometrica*, 62, 853-880.

2. Hernandez A. and M.S. Santos (1996), "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets", *Journal of Economic Theory*, 71, 102-130.

3. Magill, M. and M. Quinzii (1996). "Incomplete Markets over an Infinite Horizon: Long-Lived Securities and Speculative Bubbles", *Journal of Mathematical Economics,* 26, 133-170.

4. Santos, M.S. and M. Woodford (1997), "Rational Asset Pricing Bubbles", *Econometrica*, 65, 19-57.

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7. Duffie, D., Geanakoplos, J., Mas-Colell, A., and A. McLennan (1994), "Stationary Markov Equilibria", *Econometrica*, 62, 745-781.

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9. Huggett, M. (1993), "The Risk-Free Rate in Heterogeneous-Agent, Incomplete-Insurance Economies", *Journal of Economic Dynamics and Control*, 17, 953-969.

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Incomplete Markets

Volume 2: Infinite Horizon Economies

This volume studies models of incomplete markets over an infinite horizon. The finite-horizon models studied in Volume 1 can be extended to an infinite horizon in two ways: the first approach retains the same finite collection of agents and extends the horizon to an open-ended future; the second introduces an infinite sequence of agents through a demographic structure in which all agents have finite lives and are replaced by their children in an infinite sequence of overlapping generations. Both models conform more closely to the world in which we live in that each day there is always a tomorrow. The first class of models, which we refer to as the *infinite horizon models*, clearly violate reality however by allowing agents to remain on stage forever, although some realism can be gained by interpreting each agent as one long family "dynasty". Such an interpretation however requires a perfect co-ordination among the different generations (e.g. children cannot renege on the debt of their ancestors) in order that they behave as a single agent, a co-ordination requirement which may seem somewhat excessive. In this respect the *overlapping generations model* (OLG) conforms much better to the real world. However infinite horizon models have a simpler structure than OLG models and more is known about them, especially for stochastic economies. They are particularly useful for studying economic properties which do not depend excessively on the heterogeneity of the agents, so that the economy can approximated by either a representative agent or by a few "representatives" from the different income and preference groups. For questions for which the presence of successive generations plays an essential role—such as social security or the use of depletable resources—the OLG model provides greater insight into the fundamental issues involved.

I. Existence of Equilibrium, Ponzi Schemes, and Asset Price Bubbles

With this in mind we focus in this volume on the the class of infinite horizon models. A few reflections may be useful to put this class of models into perspective. Consider first the issue of an agent's preferences. Time should not matter at all for an infinite-lived agent since at every date an infinite future always lies ahead. As John Rae (1834) and Frank Ramsey (1928) pointed out forcefully, it would be indefensible for such an agent to discount the future.

The economic model however has a hard time coping with such a purist model of an infinite-lived agent: as Bewley (1972) showed in his extension of the Arrow-Debreu theory to an infinite horizon, it is a prerequisite for obtaining an equilibrium with well-defined present-value prices that agents' preference orderings exhibit "impatience", expressed formally by the condition that the preference orderings be continuous in the Mackey topology. All the papers in this volume either explicitly or implicitly make this assumption on agents' preferences. In simple language this assumption implies that a promise of one unit of consumption at any date-event in the future is worth less than one unit of consumption today. The hypothesis of impatience is an inevitable compromise that we are forced into when working with the "thought experiment" of infinite-lived agents: agents must put more weight on the present than on events in the future if consumptions streams, and more generally annuities and infinite-lived securities such as equity, are to have finite present values. In the real world discounting by agents is fundamentally a reflection of the finiteness and uncertainty of life, and the model requires that we retain this discounting when we use the idealization of infinite-lived "dynasties".

In the finite-horizon models studied in Volume 1, an agent is not permitted to end up with debt at the terminal date: this imposes a limit on how much the agent can borrow at preceding dates. When the horizon is extended to infinity this terminal condition disappears and a new possibility arises: an agent can borrow at some date and postpone the repayment of the debt to the next period by borrowing the amount of the principal plus interest, rolling the progressively accumulating debt for one period to the next ad infinitum. If such a possibility is present then there will be no solution to an agent's choice problem since there is no limit to the extent to which borrowing can be used to finance more consumption. A portfolio strategy in which an agent borrows at some date and postpones the repayment of the debt indefinitely by rolling it over at every subsequent date is called a *Ponzi scheme*. To obtain an equilibrium in the infinite-horizon model agents must be prevented from entering into Ponzi schemes. As we shall see there are a number of ways in which this can be done, but all methods have in common the property that they limit the extent to which agents can accumulate debts.

Several papers (Magill-Quinzii (MQ for short) (1994, Chapter 1), (1996, Chapter 3), Hernandez-Santos (HS) (1996, Chapter 2), Levine-Zame (1996)) have studied conditions for existence of a sequential equilibrium with an incomplete financial structure, for the class of economies that Bewley (1972) originally studied using the Arrow-Debreu complete market structure. This class of economies is the natural extension to an infinite horizon of the economies studied in Volume 1. The time-uncertainty structure is represented by an event tree $\mathbb{D}$ where each date-event (node) $\xi \in \mathbb{D}$ is followed by a finite number of contingencies in the next period. There are I infinitely-lived agents consuming L perishable goods at each node and each agent chooses a consumption plan which lies in the non-negative orthant of the space $l_\infty = \left\{ x \in \mathbb{R}^{\mathbf{D} \times L} \,\middle|\, \sup_{(\xi,\ell) \in \mathbf{D} \times L} x(\xi)_\ell < \infty \right\}$ consisting of bounded sequences. Each agent has an initial endowment $\omega^i \in l_\infty^+$ and chooses a consumption plan $x^i \in l_\infty^+$. With the Arrow-Debreu structure of markets studied by Bewley (1972), at the initial date there are contingents markets for the delivery of each good at each date-event and an agent faces a single budget constraint $P(x^i - \omega^i) = \sum_{\xi \in \mathbf{D}} P(\xi)(x^i(\xi) - \omega^i(\xi)) = 0$, $P(\xi) = (P_\ell(\xi))_{\ell \in L}$ being the vector of prices at date 0 for commodities deliverable at node ξ. In all the papers in this Volume, agents trade sequentially on a system of spot and financial markets at each date-event $\xi \in \mathbb{D}$. Each agent thus has a sequence of budget constraints, one for each date-event

$$p(\xi)(x^i(\xi) - \omega^i(\xi)) = R(\xi) z^i(\xi^-) - q(\xi) z^i(\xi), \qquad \xi \in \mathbb{D}$$

where $p(\xi) \in \mathbb{R}^L_+$ is the vector of spot prices for the goods, $R(\xi)$ is the matrix of payoffs of the securities (traded at the preceding node ξ^- and sold at node ξ if long-lived), $q(\xi)$ is the vector of prices of the securities and $z^i(\xi)$ is the portfolio of agent i at node ξ.

To prove existence of an equilibrium in this sequential model, two conditions need to be added to Bewley's assumptions: the assumption of impatience, formalized by the assumption of continuity of the preferences in the Mackey or the product topologies (the two topologies coincide on bounded sets) needs to be strengthened to an assumption of *uniform impatience*, and a condition preventing Ponzi schemes needs to be added to the node-by-node budget constraints. The papers mentioned above differ in the way they express these two conditions. The condition of uniform impatience is essentially a property of stationarity of preferences, in that it requires that the trade-off between one unit of consumption at date t and consumption in the future has a lower bound independent of t. The condition of uniform impatience ensures that the security prices are uniformly bounded, and is used to prove that the equilibria of finite-horizon truncated economies converge.

To express the no-Ponzi scheme condition, MQ require that on the subtree originating from each date-event ξ the present value of consumption at node ξ is equal to the accumulated wealth of the agent (possibly negative) augmented by the present value of the agent's future income. This is equivalent to requiring that on the subtree following node ξ the present value of the agent's debt at date T tends to zero as T tends to infinity, and thus limits the amount of debt which can be incurred. Since, with incomplete markets, present values are not uniquely defined (there are many present-value prices compatible with the equilibrium prices of the securities, see Volume 1) MQ use the personal valuation of an agent to evaluate the present values of the agent's consumption, income and debts. This leads to a convenient way of extending Bewley's approach to the setting of sequential and incomplete markets, although it is perhaps difficult to interpret this as an enforcement mechanism. However once the existence of equilibrium is established, MQ show that the debts $q(\xi)z^i(\xi)$ of the agents are uniformly bounded below, so that the same equilibrium can be obtained by imposing a uniform bound M on the debts of every agent, and this bound will never bind in equilibrium. This is the simplest way of imposing a debt constraint which is easy to monitor and does not add an additional "imperfection" to the equilibrium, over and above the incompleteness of markets.

Hernandez-Santos use essentially the same approach but with a different way of evaluating the present value of an agent's future endowment: at each node they consider the infimum of the present values of the agent's future endowment, for all the node price processes compatible with the security prices, and require that at each node the agent's debt is less than this infimum. They show that this infimum as a nice interpretation: it is the largest debt that an agent can pay back in finite time T, for some $T < \infty$. Levine-Zame (1996) use the same constraint of limiting debt to what can be repaid in finite time, without linking it to the present value of future income. Under the assumption of uniform impatience all these different ways of imposing debt constraints so as to avoid Ponzi schemes lead to the same equilibria, a property emphasized by Florenzano-Gourdel (1996).

The existence results mentioned above apply to economies in which securities are short lived—they mature (expire) at the date immediately following their date of issue—and pay amounts of the same good (numeraire securities) or amounts of money (nominal securities). As soon as securities are long-lived (they last for two or more periods after their date of issue), or promise to deliver the spot-market value of bundles of goods, there can be changes in the rank of the payoff matrix $R(\xi)$

and, as we saw in Volume 1, this can create discontinuities in agents' demands and possible non-existence of equilibrium, even in finite horizon economies. Thus HS (Chapter 2) and MQ (Chapter 3) use the approach developed for finite horizon economies and begin by proving existence of a pseudo-equilibrium, in which the subspaces of income transfers are of fixed dimension, to avoid discontinuities. They then study when the pseudo-equilibria are "true" equilibria. The results are weaker than in the finite-horizon case: Chapters 2 and 3 prove existence of equilibria of economies with long-lived securities for a dense subset of endowment-payoff parameters, but a property akin to the full-measure property of generic sets of parameters, for which equilibria exist in the finite-horizon case, is missing.[1]

Security prices have many interesting properties and one which is dramatically altered by the presence of an open-ended future is whether or not the price of a security coincides with the present value of its dividends. This has been the subject of extensive discussions in the macro literature—often with the idea that the presence of a "speculative bubble" which disconnects the security price from the fundamental value of its dividends may lead to a misallocation of resources. So it became of considerable interest to open up this discussion again when this class of infinite-horizon sequential models were systematically analyzed. One especially innovative paper here is that by Santos and Woodford (SW for short) (1996, Chapter 4) who lay out a framework for analyzing the conditions under which bubbles may or may not appear in the sequential equilibria of a general class of economies, which include both infinite horizon and OLG economies with possibly incomplete financial markets. In Chapter 3, MQ analyze bubbles in the more limited setting of the infinite horizon model, under the assumption of uniform impatience. These two papers show that bubbles can affect the allocation of resources: however it is easy to construct examples in which bubbles are "helpful" rather than "harmful" because they serve to augment the span of the financial markets.

Whether or not equilibrium prices of securities can have bubbles depends importantly on whether the securities are in positive or zero net supply. Securities in positive supply model ownership of income (dividend) streams created by productive assets such as firms, land or other durable goods: the most familiar class of securities in positive supply are the equity contracts of corporations traded on the stock market. Securities in zero net supply model borrowing and lending and the large class of derivative securities such as options—although the maturity dates of such

[1] To our knowledge these existence results have not yet been improved.

contracts are usually relatively short. Only in exceptional cases, such as for consols and perpetual warrants, are they infinitely lived.

To give a slightly more precise overview of the contributions of the papers in Part I, we will use the notation of Magill-Quinzii in Chapters 1 and 3, which we began to introduce above. As we saw in Volume 1 a fundamental property of equilibrium security prices is that they do not offer arbitrage opportunities: at each node ξ of the event-tree $\mathbb{D}$ it is not possible to find a portfolio $z(\xi)$ such that

$$q(\xi)z(\xi) \leq 0 \qquad R(\xi')z(\xi) \geq 0, \quad \forall\, \xi' \in \xi^+, \text{ with at least one strict inequality}$$

where ξ^+ denotes the set of nodes immediately following node ξ (the successors of ξ), and $R(\xi')$ denotes the payoffs of the securities at the successor node ξ': for a short-lived security j, $R^j(\xi')$ is just the *dividend* paid by the security, which we denote by $V^j(\xi')$; that is, $R^j(\xi') = V^j(\xi')$. For a long-lived security j, the payoff includes the resale price $q^j(\xi')$ of the security, i.e. $R^j(\xi') = V^j(\xi')+q^j(\xi')$.

Absence of arbitrage implies that there is a strictly positive vector of node (present-value) prices $\pi = (\pi(\xi), \xi \in \mathbb{D})$ for income at each node such that

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')R(\xi'), \qquad \forall \xi \in \mathbb{D} \tag{1}$$

In finite horizon economies since security prices are zero at the terminal date, successive substitution of (1) leads to the valuation formula

$$q^j(\xi) = \sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\pi(\xi')}{\pi(\xi)} V^j(\xi')$$

where $\mathbb{D}(\xi)$ denotes the subtree originating at node ξ and $\mathbb{D}^+(\xi)$ denotes the nodes strictly after node ξ, so that the price of an asset at any node is equal to the present value (at this node) of its future dividends, the so-called *fundamental value* of the asset. Since HS and MQ establish existence of equilibrium for an infinite horizon economy by taking limits of equilibria of truncated (finite-horizon) economies, in such an equilibrium the price of an infinite-lived security always coincide with the present value of its dividends. However this does not preclude the existence of other equilibria, in which the price of an asset exceeds its fundamental value, the difference being called a *bubble*. SW present a systematic study of the conditions under which such bubbles are possible in equilibrium.

One of the difficulties presented by a study of the pricing of income streams over an infinite horizon when markets are incomplete is that, for a given process of security prices, there are many node price processes $\pi = (\pi(\xi), \xi \in \mathbb{D})$ which satisfy the no-arbitrage pricing relations (1). Thus an income stream may have many "fundamental value". SW consider the upper and lower bounds for the fundamental values of a bounded income stream, and provide a natural economic interpretation of these bounds. To show how these bounds are obtained, suppose that a node ξ an agent owns an asset which will provide a stream of dividend income $(y(\xi'),\ \xi' \in \mathbb{D}^+(\xi))$ on the subtree $\mathbb{D}^+(\xi)$ consisting of all nodes in the tree strictly after node ξ. What is the most that an agent can borrow at ξ against this asset, i.e what is the largest possible value of $-q(\xi)z(\xi)$ if the portfolio strategy z needs to satisfy the solvency constraints

$$y(\xi') + R(\xi')z(\xi'^-) - q(\xi')z(\xi') \geq 0, \qquad \forall \xi' \ \in \ \mathbb{D}^+(\xi) \tag{2}$$

and if the debt needs to be fully reimbursed in finite time, i.e. there exist $T > t(\xi)$ such that

$$q(\xi')z(\xi') \geq 0, \qquad \forall \xi' \in \mathbb{D}^+(\xi),\ \text{with } t(\xi') > T \tag{3}$$

Multiplying the solvency constraint at ξ' by $\pi(\xi')$ and summing over the subtree $\mathbb{D}^+(\xi)$, using (1) and (3), gives

$$-q(\xi)z(\xi) \leq \sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\pi(\xi')}{\pi(\xi)} V^j(\xi')$$

This relation must hold for all allowable portfolio strategies Z_ξ beginning at node ξ, and all no-arbitrage node prices NA_ξ on the subtree $\mathbb{D}(\xi)$, so that

$$\text{sup}_{z \in Z_\xi} - q(\xi)z(\xi) \leq \text{inf}_{\pi \in NA_\xi} \frac{\pi(\xi')}{\pi(\xi)} V^j(\xi') \tag{4}$$

Thus the smallest present value of the income stream is bounded below by the largest amount that an agent can borrow against this income stream, being sure to reimburse in finite time. This leads HS in Chapter 2 to impose the condition that the debt of an agent at each node does not exceed the infimum present value of the agent's endowment stream as the borrowing constraint for preventing Ponzi schemes.

Suppose now that we consider an agent who, instead of owning the income stream y, wants to acquire it. How much will it cost? The only way to acquire the income stream y is to find a portfolio $z \in Z_\xi$ which super-replicates y in the sense that

$$R(\xi')z(\xi'^-) - q(\xi')z(\xi') \geq y(\xi'), \qquad \forall\, \xi' \in \mathbb{D}^+(\xi) \tag{5}$$

Multiplying (5) by $\pi(\xi')$ and summing over the subtree $\mathbb{D}^+(\xi)$ gives

$$\sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\pi(\xi')}{\pi(\xi)} y(\xi') \leq q(\xi)z(\xi)$$

for all $z \in Z_\xi$ and all $\pi \in NA_\xi$, so that

$$\sup_{\pi \in NA_\xi} \sum_{\xi' \in \mathbb{D}^+(\xi)} \frac{\pi(\xi')}{\pi(\xi)} y(\xi') \leq \inf_{z \in Z_\xi} q(\xi)z(\xi)$$

The largest present value of the income steam y is bounded above by the smallest cost of a portfolio which super-replicates y in the sense of (5).

Santos-Woodford's analysis in Chapter 4 is based on the following property. Suppose that the price of a security in positive supply (which we can normalize to 1) has a bubble at node ξ, i.e. the price exceeds the present value of the dividends under a no-arbitrage node price process π

$$\pi(\xi)q^j(\xi) > \sum_{\xi' \in \mathbb{D}^+(\xi)} \pi(\xi')V^j(\xi')$$

Since by integrating the no-arbitrage relation (1) up to date T, the price of the security is equal to the present value of the dividends up to date T plus the capital value at date T

$$\pi(\xi)q^j(\xi) > \sum_{\xi' \in \mathbb{D}^+(\xi), t(\xi') \leq T} \pi(\xi')V^j(\xi') + \sum_{\xi' \in \mathbb{D}(\xi), t(\xi') = T} \pi(\xi')q^j(\xi') \tag{6}$$

it follows that $\sum_{\xi' \in \mathbb{D}(\xi), t(\xi')=T} \pi(\xi')q^j(\xi')$ does not tend to zero when T tends to infinity. If, on the other hand, the present value of aggregate output is finite

$$\sum_{\xi' \in \mathbb{D}(\xi)} \pi(\xi')p(\xi')\omega(\xi') < \infty$$

where $\omega(\xi)$ is the aggregate endowment (consumption) at node ξ and $p(\xi)$ the vector of spot prices, then $\sum_{\xi' \in \mathbb{D}(\xi), t(\xi')=T} \pi(\xi')p(\xi')\omega(\xi')$ tends to zero when T tends to infinity (since the series converge) while $\sum_{\xi' \in \mathbb{D}(\xi), t(\xi')=T} \pi(\xi')q^j(\xi')$ does not. It follows that infinitely often on the subtree originating at ξ the agents must hold a portfolio which is worth more than the total consumption in the economy and, unless agents value consumption "at infinity", this is incompatible with the maximizing behavior of the agents.

Thus the main result of SW in Chapter 4 is that if at each node there is a portfolio of marketed securities which super-replicates the value of aggregate consumption in the sense of (5)—which implies that the value of aggregate consumption is finite—then in equilibrium there is a no-arbitrage process of node prices such that all infinite-lived securities in positive supply are priced at fundamental value. They also show that under the assumption of uniform impatience the fundamental value is the same for all no-arbitrage processes of node prices. Note that the assumption that there is a traded portfolio which super-replicate aggregate output will be satisfied if the net dividend (gross profit minus gross investment) of the firms traded on the stock market is bounded below by a positive fraction of GDP, a property which apparently seems to be verified empirically (Abel et all (1989)).

The merit of the Santos-Woodford analysis is its generality. A simpler but more limited result is given by MQ in Chapter 3. Agents are assumed to be infinitely lived and to satisfy uniform impatience. MQ show that the impatience property implies that the value $q(\xi)z^i(\xi)$ of an agent's portfolio is bounded above $(q(\xi)z^i(\xi)_{\xi\in\mathbb{D}} \in \ell_\infty$: impatient agents buy consumption rather than letting their wealth accumulate to infinity. If $\sum_i z^i(\xi) = 1$, then $(q(\xi))_{\xi\in\mathbb{D}}$ must be in ℓ_∞. Impatience also implies that the personalized node prices of the agents are summable: the marginal rate of substitution between income now and income in the distant future tends to zero sufficiently fast to imply $\sum_{\xi\in\mathbb{D}} \pi^i(\xi) < \infty$. But then the pricing relation (6) with the personalized node prices of agent i

$$\pi^i(\xi)q^j(\xi) = \sum_{\xi'\in\mathbb{D}(\xi),t(\xi')\leq T} \pi^i(\xi')V^j(\xi') + \sum_{\xi'\in\mathbb{D}(\xi),t(\xi')=T} \pi^i(\xi')q^j(\xi')$$

implies that the security price equals the present value of the dividends with the no-arbitrage node price π^i: since $q^j \in \ell_\infty$ and $\pi^i \in \ell_1$ the capital value term at date T tends to zero when T tends to infinity. Thus, at least for the personalized node prices of the agents, there cannot be a bubble in the price of a security in positive supply.

The argument that the portfolio wealth (qz^i) of an agent is bounded cannot be applied to eliminate the possibility of bubbles in the prices of securities in zero net supply, since $q^j(\xi)$ may tend to ∞ if $z^i_j(\xi)$ tends to zero sufficiently fast. For infinite-lived securities in zero net supply, MQ show that is always possible to add a bubble component to an equilibrium price process and the resulting new price process is still an equilibrium price. It may be that an equilibrium, in which a security in zero net supply has a bubble, can not be obtained if securities are priced at fundamental value, because this results in a smaller span of the markets. Thus, contrary to popular wisdom, a (rational) bubble can augment the possibility of intertemporal trades and improve social welfare. However since securities in zero net supply are rare in the real world (bonds and derivatives securities typically have finite maturity dates) the result that bubbles can increase the span of the market and help the economy seems to be more of a more a theoretical possibility than a practical way of increasing the possibilities for risk sharing in the economy.

The overall message that emerges from this analysis of “rational bubbles” is that if in practice we observe situations where the prices of assets seem far out of line with a reasonable estimate of the present value of their dividends, then the origin of such phenomena comes from sources which cannot be explained by the correct anticipations equilibria of the models studied above. We must look outside the framework of the rational expectations models to find a reasonable explanation of “speculative bubbles”.

In the papers studied so far (Chapters 1-4), agents never default on their short-term obligations: if an agent borrows at some date t, then he pays back the principal and interest at $t+1$, but he may take out a new loan at $t+1$ to make these payments and in this way postpone the repayment of his debt. If an agent can infinitely often postpone the repayment of his debt, then the agent has in essence found a way of never repaying his debt, or of defaulting “at infinity”—and this is what a Ponzi scheme is about. Thus the conditions discussed in Chapters 1-5 are essentially conditions which prevent default at infinity.

When agents are prevented from defaulting at infinity, they may have an incentive to default on their short-term obligations. The phenomenon of default makes financial markets fragile, and a wide array of institutional arrangements have been put in place to ensure that agents have an incentive to repay, and, in case where there is default, to organize how the default is shared among the lenders. An interesting point made by Araujo, Pascoa and Torres-Martinez (APTM) in Chapter 5 is that institutions which serve to limit the possibility of default on short-term obligations may at the same time prevent agents from entering into Ponzi schemes.

Every contractual exchange on a financial market is in essence akin to a loan: there is a benefit (the loan) and a cost (reimbursement). Seeking default is like trying to get the benefit without incurring the cost—and if mutually beneficial exchange is to survive, society must put into place "penalties" of some kind to ensure that there is always a quid pro quo. Thus if agents are assumed to be opportunistic and to pay for their loans only if it is in their interest to do so, then they must incur a penalty if they default—otherwise they would systematically choose to default, and mutually beneficial intertemporal exchange would be impossible.

There are essentially three approaches that have been proposed for modeling penalties for default in general equilibrium. The first is that of the paper of Dubey-Geanakoplos-Shubik in Volume 1 which assumes that if agents default on their obligations they incur a utility penalty proportional to the amount of the default. This approach can be viewed as a short-cut which seeks to avoid modeling the details of the institutional arrangements which make it undesirable for an individual agent to default—the stylized example in ancient times being the debtor prison.

The second approach assumes that every short position in a contract must be covered by collateral, i.e. an agent must either purchase or have on hand an amount of a durable good which can be appropriated by the creditors if the agent defaults on his obligation. This is the approach which was introduced by Geanakoplos-Zame (1995, 2002) and is studied in an infinite horizon setting by Araujo-Pascoa-Torres-Martinez (2002, Chapter 5, APTM for short).

Enforcement of penalties or appropriation of collateral are effective methods of discouraging default if there is a legal system which can, at a reasonable cost, enforce such penalties. If either the legal system does not exist or the cost of enforcement is high, other types of penalties must be found for discouraging agents from defaulting on their loans: this situation is of particular relevance for the debt of sovereign nations whose contractual obligations are not enforceable by international law. Eaton and Gersovitz (1981) pointed out that the penalty which applies to sovereign nations which default on their loans is exclusion from subsequent participation in the international financial markets. Although in practice the exclusion is temporary, they formalized an equilibrium concept in which default is followed by permanent subsequent exclusion from the market. This idea was further developed in a general equilibrium setting by Kehoe-Levine (1993) and Alvarez-Jermann (2000, Chapter 6). This third approach for modeling the penalty for default is interesting because it leads to endogenous debt constraints, or more bluntly credit limits, which express the maximum amount that can be safely lent to an agent at each date-event when the incentive to repay is given by the threat of exclusion.

Consider first the APTM paper (Chapter 5) which studies an institutional setting where the provision for default consists of the requirement that a borrower either purchase or have in hand collateral goods which cover his short position by ensuring a minimum payment on his obligation. The paper shows that this collateral constraint, when enforced at each date-event, is sufficient to prevent agents from entering into Ponzi schemes.

In a model with collateral some of the goods must be durable. We say that good ℓ is *durable* if it can be carried forward from one period to the next, and one unit at date t gives α_ℓ units at date $t+1$, with $0 < \alpha_\ell \leq 1$; the good is *perishable* if $\alpha_\ell = 0$. APTM assume that all durable goods depreciate, $\alpha_\ell < 1$ for all ℓ, so that the amount of each good available for consumption cannot grow too fast. Let $C^j \in \mathbb{R}^L$ denote the collateral requirement for security j. If an agent sells one unit at node ξ with a promise to pay $V^j(\xi')$ at each successor $\xi' \in \xi^+$, and if there is no other penalty for defaulting than the seizure of the collateral, then the actual payment of an agent at node ξ' will be

$$\min\{V^j(\xi'),\, p(\xi')[\alpha]C^j\}$$

where $[\alpha]$ is the diagonal matrix of depreciation factors and $p(\xi')$ is the vector of spot prices for the goods at node ξ'. If $p(\xi')[\alpha]C^j < V^j(\xi')$, the agent defaults on the contract and the creditors appropriate the collateral: this is anticipated by all agents in equilibrium. If all contracts on which agents can take short positions (borrow) have a strictly positive collateral requirement, and if the aggregate endowment is bounded over the event-tree, then the short positions that agents can take are limited by the scarcity of the available collateral. The proof of this property rests on the following "no-arbitrage" condition

$$q^j(\xi) \leq p(\xi)C^j \tag{7}$$

The price of the contract secured by the collateral C^j can not exceed the purchase price $p(\xi)C^j$ of the collateral. Otherwise, if $q^j(\xi) > p(\xi)C^j$, an agent could sell one unit of the contract, receive $q^j(\xi)$, buy the corresponding collateral for the cost $p(\xi)C^j$, making a profit of $q^j(\xi) - p(\xi)C^j$. At the next date, if $V^j(\xi') < p(\xi')[\alpha]C^j$ the agent can sell the depreciated collateral, pay $V^j(\xi')$ and pocket the difference; otherwise the agent defaults and the creditors appropriate the collateral. This is clearly an arbitrage opportunity with a sure gain at node ξ and no losses at the successors.

Thus if $\phi^i_j(\xi)$ denotes the short position of agent i on security j at node ξ, then summing over the agents and using (7) gives

$$q^j(\xi)\sum_{i=1}^{I}\phi^i_j(\xi) \leq p(\xi)C^j\sum_{i=1}^{I}\phi^i_j < M \tag{8}$$

for some $M > 0$, since the amounts of the collateral goods are bounded and the spot price vector $p(\xi)$ can be normalized. Thus the amount of debt $q^j(\xi)\phi^i_j(\xi)$ of each agent on each security is uniformly bounded over the event-tree and, as a result, no agent can indefinitely roll over his debt to infinity—an assumption of impatience ensuring that the interest rates are positive on average so that any Ponzi scheme would involve an unbounded debt.

For this reasoning to work, the appropriation of the collateral must be the only penalty for default. Pascoa-Seghir (2005) show that if in addition to the appropriation of collateral there is a sufficiently large utility penalty for default, then the collateral requirement no longer precludes rolling over debt to infinity. For the inequality (7) is no longer necessary since the operation of short selling the security and buying the collateral described above may involve utility penalties and may not constitute an arbitrage. In the limit, when the utility penalties tend to infinity, the model essentially reduces to the models studied in Chapters 1-4, where some form of debt constraint or transversality condition is required to eliminate Ponzi schemes.

In settings where it is not practical to impose collateral constraints, either because there is no readily available collateral which can be seized as in the case of credit card loans, or because there is no readily available enforcement mechanism, as in the case of loans to sovereign nations where there is no international court of law, the third type of penalty, exclusion from the market in case of default, becomes a relevant penalty to consider. A concept of equilibrium based on this approach is studied by Alvarez and Jermann (AJ for short) in Chapter 6 based on earlier work by Eaton and Gersovitz (1981), Kehoe and Levine (1993) and Kocherlakota (1996). The analysis of these papers rests on three strong assumptions:

1. Lenders can co-ordinate to enforce the exclusion of any agent who defaults from all subsequent participation in the financial markets, either as a lender or as a saver, forever after.

2. Lenders are assumed to know the characteristics of borrowers (they are thus presumably intermediaries) and can design agent-specific credit limits at every node of the event-tree equal (or marginally inferior) to the minimum amount of consumption that would compensate the agent for

perpetual exclusion from the market.

3. Financial markets are complete: agents can insure all idiosyncratic risks as well as share the aggregate (macroeconomic) risks.

The subsequent literature has attempted to weaken the assumptions 1 and 2, but assumption 3, which greatly simplifies the analysis has always been retained. Since in these models markets are assumed to be complete, default has no beneficial role to play and only appears as a cost induced by imperfect enforcement of contracts. This gives a different view of default than the models with incomplete markets, in which some amount of default is welfare improving for the economy since it increases the possibilities of risk sharing: see Dubey-Geanakoplos-Shubik (2005) in Volume I.

To understand how the credit limits are derived in Chapter 6, a little notation will be helpful. For any node ξ in the event-tree $\mathbb{D}$, let $\mathbb{D}(\xi)$ denote the subtree from node ξ on and let $\beta(\xi)$ denote the branching number at ξ, namely the number of elements in ξ^+ (the set of successors of ξ). AJ study the basic one-good financial model where at each node $\beta(\xi)$ Arrow securities are traded: Arrow security (ξ, ξ') is issued at node ξ, promises to pay one unit of income at node ξ' and has the price $q_{\xi'}(\xi)$. If $z^i_{\xi'}(\xi)$ is the amount of this security purchased by agent i, then the basic budget equation linking the agent's trade at ξ to his consumption $x^i(\xi')$ at the immediate successor ξ' is given by

$$x^i(\xi') = \omega^i(\xi') + z^i_{\xi'}(\xi) - \sum_{\xi'' \in \xi'^+} q_{\xi''}(\xi') z^i_{\xi''}(\xi') \tag{9}$$

where $z^i_{\xi''}(\xi')$ is the amount of security (ξ', ξ'') purchased at node ξ'. The budget equation (9) for node ξ' assumes that agent i reimburses $z^i_{\xi'}(\xi)$ if this is a short position. If the agent defaults, then $x^i(\xi'') = \omega^i(\xi'')$ on the whole subtree $\mathbb{D}(\xi)$: the agent is excluded from financial markets and lives in autarchy, consuming his endowment thereafter.

The analysis of AJ is simplified by the assumption that agettns have expected-utility preferences: there exist a concave increasing utility function $v^i : \mathbb{R} \to \mathbb{R}$ such that

$$u^i(x^i) = E\left(\sum_{t=0}^{\infty} \delta_i^t v^i(x^i_t)\right) = \sum_{\xi \in \mathbb{D}} \delta_i^{t(\xi)} \rho(\xi) v^i(x^i(\xi)), \quad i \in I$$

The assumption of separability makes it easy to evaluate the utility of an agent from consumption restricted to a subtree $\mathbb{D}(\xi)$

$$u^i_{\mathbb{D}(\xi)}(x^i) = \sum_{\xi' \in \mathbb{D}(\xi)} \delta_1^{t(\xi')} \rho(\xi') v^i(x^i(\xi'))$$

Each agent chooses a trading strategy $z^i = (z^i_{\xi'}(\xi), \xi' \in \xi^+, \xi \in \mathbb{D})$ for the Arrow securities over the event-tree. Suppose the agent's trading strategy is constrained by a vector of credit limits $b^i = (b^i_{\xi'}(\xi), \xi' \in \xi^+, \xi \in \mathbb{D})$ theset of feasible trading strategies for agent i is given by

$$Z(b^i) = \left\{ z^i \in \prod_{\xi \in \mathbb{D}} \mathbb{R}^{\beta(\xi)} \,\middle|\, z^i_{\xi'}(\xi) \geq -b^i_{\xi'}(\xi),\ \forall\ \xi' \in \xi^+,\ \forall\ \xi \in \mathbb{D} \right\}$$

Consider a date-event $\xi \in \mathbb{D}$ and let's see if agent i would or would not like to default at this node. Let η denote the debt or credit inherited from the agent's position at node ξ^-. If the agent does not default at ξ or any of its successors, then the available consumption streams on the subtree $\mathbb{D}(\xi)$ are

$$\mathbb{B}^i_{\mathbb{D}(\xi)}(\omega^i, \eta, b^i, q) = \left\{ x^i \in \mathbb{R}^{\mathbb{D}(\xi)}_+ \,\middle|\, \begin{array}{l} x^i(\xi) = \omega^i(\xi) + \eta - q(\xi) z^i(\xi) \\ x^i(\xi') = \omega^i(\xi') + z^i(\xi'^-) - q(\xi') z^i(\xi') \\ \text{for all } \xi' \in \mathbb{D}(\xi), \text{ for some } z^i \in Z(b^i) \end{array} \right\}$$

Let $x^{i*}(\eta)$ denote the agent's most preferred consumption stream in $\mathbb{B}^i_{\mathbb{D}(\xi)}$, namely the one which maximizes $u^i_{\mathbb{D}(\xi)}$ over $\mathbb{B}^i_{\mathbb{D}(\xi)}$. If

$$u^i_{\mathbb{D}(\xi)}(x^{i*}(\eta)) < u^i_{\mathbb{D}(\xi)}(\omega^i)$$

then the agent will default on the debt η: of course the inequality can only occur if $\eta < 0$. Thus in order that the debt η does not induce default at node ξ we must have $u^i_{\mathbb{D}(\xi)}(x^{i*}(\eta)) \geq u^i_{\mathbb{D}(\xi)}(\omega^i)$. Since $\eta > \eta'$ implies $u^i_{\mathbb{D}(\xi)}(x^{i*}(\eta)) \geq u^i_{\mathbb{D}(\xi)}(x^{i*}(\eta'))$ the maximum amount of debt $b^i_\xi(\xi^-)$ that can be permitted at the predecessor ξ^-—the agent's *credit limit* for security (ξ^-, ξ)— is defined by

$$u^i_{\mathbb{D}(\xi)}(x^{i*}(-b^i_\xi(\xi^-))) = u^i_{\mathbb{D}(\xi)}(\omega^i) \tag{10}$$

(10) is just a recursive system of equations across the event tree: when the credit limits are known for all $\tau \geq t$ then (10) defines the credit limits for date $t-1$.

An *equilibrium with credit limits* $((\bar{x}, \bar{z}, \bar{b}), \bar{q})$ with initial positions $(z^i_0)_{i \in I}$ is a vector of actions $(\bar{x}^i, \bar{z}^i)_{i \in I}$, credit limits $(b^i)_{i \in I}$ and security prices $\bar{q}$ such that

(i) for each agent $i \in I$, $(\bar{x}^i, \bar{z}^i)$ maximizes $u^i(x^i)$ over $\mathbb{B}^i_{\mathbb{D}(\xi_0)}(\omega^i, z^i_0, \bar{b}^i, \bar{q})$ and $\bar{z}^i$ finances $\bar{x}^i$

(ii) for each agent $i \in I$ and each node $\xi \in \mathbb{D}$, the credit limit is the largest compatible with the agent not defaulting

$$\max \left\{ u^i_{\mathbb{D}(\xi)}(x^i) \,\middle|\, \xi \in \mathbb{B}^i_{\mathbb{D}(\xi)}(\omega^i, -\bar{b}^i_\xi(\xi^-), \bar{b}^i, \bar{q}) \right\} = u^i_{\mathbb{D}(\xi)}(\omega^i)$$

(iii) $\sum_{i\in I} \bar{z}^i = 0$: financial markets clear

This is a standard financial market equilibrium over an infinite horizon with debt constraints, with the additional proviso that the debt constraints or assigned credit limits $\bar{b} = (\bar{b}^i)_{i\in I}$ ensure that no agent wants to default in equilibrium at any date event, if the penalty for default is permanent exclusion from the financial markets. AJ call this an equilibrium with solvency constraints. Their analysis focuses on two types of equilibria:
(a) equilibria in which the present value of the aggregate endowment is finite
(b) the no-trade equilibrium (autarchy)

Equilibria of type (a) can also be viewed as constrained Arrow-Debreu equilibria of the following type. A pair $((\bar{x}^i)_{i\in\ I}, \bar{P})$ is a *constrained Arrow-Debreu equilibrium* if

(i) for each $i \in I$, $\bar{x}^i$ maximizes $u^i(x^i)$ subject to

(α) $\bar{P}(x^i - \omega^i) = 0$

(β) $u^i_{\mathbb{D}(\xi)}(x^i) \geq u^i_{\mathbb{D}(\xi)}(\omega^i), \ \forall\ \xi \in \mathbb{D}$

(ii) $\sum_{i\in I}(\bar{x}^i - \omega^i) = 0$

Implicit in this definition is the assumption that an agent can at any node $\xi \in \mathbb{D}$ choose to opt out of the promises made at date 0, reverting to autarchy forever after. This is the concept of equilibrium which has been studied by Kehoe-Levine (1993, 2001).

A constrained Arrow-Debreu equilibrium is a standard Arrow-Debreu equilibrium if the consumption sets of the agents are defined by

$$X^i = \left\{ x^i \in \mathbb{R}^{\mathbb{D}}_+ \;\middle|\; u^i_{\mathbb{D}(\xi)}(x^i) \geq u^i_{\mathbb{D}(\xi)}(\omega^i),\ \forall\ \xi \in \mathbb{D} \right\}$$

Thus the First and Second Welfare Theorems hold relative to the constrained feasible allocations $\mathcal{F} = \{x \in \prod_{i\in I} X^i \,|\, \sum_{i\in I}(x^i - \omega^i) = 0\}$. Although existence of constrained AD equilibria can be established indirectly by exploiting the existence of constrained Pareto optima and the Second Welfare Theorem (given continuity assumptions in an appropriate topology for the preferences), a direct proof of existence for an arbitrary distribution of initial endowments in $\ell_\infty(\mathbb{D})$ presents problems. For the standard method of establishing existence in infinite horizon economies is to use limits of equilibria in finite horizon truncations of the economy (see Chapters 1,2,3,5). But in a finite horizon economy, for any node ξ at the terminal date T, all agents with a planned consumption $x^i(\xi) < \omega^i(\xi)$ will default since they have a higher utility by consuming their endowment. Thus there will not be any trade at date 0 for delivery of income at date T. But then for any node ξ' at date $T-1$, all agents with $x^i(\xi') \leq \omega^i(\xi')$ will default since they consume their endowment at date T. By backward induction there can not be trade at any date.

Thus the concept depends fundamentally on the fact that at any date there will be a future. To be effective, the threat of exclusion requires that at every date the agent foresees the opportunity of beneficial trades into the future: exclusion is exclusion from the benefits of future trades. To circumvent this difficulty for proving existence via truncated economies, Kehoe and Levine assume the existence of collateral: such an assumption is however an awkward foundation for a concept of equilibrium which seeks to explain how the threat of exclusion permits markets for unsecured loans to function.

To see that an equilibrium of type (a) in Chapter 6 can be transformed into a constrained AD equilibrium, consider any node ξ and let $\xi_0 < \xi_1 < \ldots < \xi_t = \xi$ denote the unique path through the event-tree leading to node ξ at date t. Since the price $q_{\xi_\tau}(\xi_{\tau-1})$ is the value at node $\xi_{\tau-1}$ of one unit of income at node ξ_τ, it follows that the product

$$P(\xi) = \prod_{\tau=1}^{t} q_{\xi_\tau}(\xi_{\tau-1}) \tag{11}$$

gives the present value at date 0 of one unit of income at node $\xi = \xi_t$. Let $P = (P(\xi), \xi \in \mathbb{D})$ denote the process of present-value prices associated with a security price process q. It is easy to see that if $((\bar{x}, \bar{z}, \bar{b}), \bar{q})$ is an equilibrium with credit limits such that $\bar{P} \sum_{i \in I} \omega^i < \infty$, then $(\bar{x}, \bar{P})$ is a constrained AD equilibrium, and thus is constrained efficient. Conversely if $(\bar{x}, \bar{P})$ is a constrained AD equilibrium, there exist portfolios, credit limits, and security prices $((\bar{z}, \bar{b}), \bar{q})$ such that $((\bar{x}, \bar{z}, \bar{b}), \bar{q})$ is an equilibrium with credit limits.

The sequential model studied by AJ has the interesting property that there is always a no-trade equilibrium $((\bar{x}, \bar{z}, \bar{b}), \bar{q}) = ((\omega, 0, 0), \bar{q})$ with credit limits identically 0 and security prices given by

$$\bar{q}_{\xi'}(\xi) = \max_{i \in I} \left\{ \frac{\delta_i \rho(\xi') v^{i\prime}(\omega^i(\xi'))}{\rho(\xi) v^{i\prime}(\omega^i(\xi))} \right\}, \quad \forall\, \xi' \in \xi^+, \quad \xi \in \mathbb{D} \tag{12}$$

The logic of the prices (12) is that even the agent who at its initial endowment is the most eager to transfer income from node ξ to node ξ' will not want to do so because the price of of the Arrow security (ξ, ξ') coincides with his marginal rate of substitution. If the implied Arrow Debreu prices $\bar{P}$ are such that the aggregate endowment has finite value, then this type (b) equilibrium is also a type (a) equilibrium and autarchy is the only possible equilibrium with credit limits. The finite present value condition $\bar{P}\sum_{i\in I}\omega^i < \infty$ expresses in a precise way the property that agents are impatient and not too desperate to trade—with a small discount factor δ_i and small variations $\omega^i(\xi') - \omega^i(\xi)$ the products of the prices (12) will be in $l_1(\mathbb{D})$ and the series $\bar{P}\sum_{i\in I}\omega^i$ will converge. These no-trade equilibria illustrate in extreme form the factors which can reduce trade with no-default credit limits: high rates of impatience and lack of variability of endowments after some date imply small credit limits and impede trade. The no-trade equilibria in which the present value of the aggregate endowment is infinite have no equivalent AD-constrained equilibria.

Equilibria with debt constraints and the threat of exclusion have the features that can be expected from equilibria with imperfect risk sharing: when compared with Arrow-Debreu equilibria (perfect risk sharing) in which an agent's consumption is correlated with aggregate output, in a constrained Arrow-Debreu equilibrium an agent consumption is typically correlated with his income (endowment). Furthermore the implied equilibrium interest rate is lower since borrowers are constrained while savers are unconstrained, so that interest rate must be lower to clear the markets. Both these properties are also typical of equilibria in models with incomplete markets, the first because risk sharing is limited by the paucity of markets, the second because the absence of risk markets gives agents an incentive to do additional precautionary savings. An advantage of the credit-limit approach over the incomplete-markets approach for generating equilibria with these properties, emphasized by Kehoe-Levine (2001), is that constrained Arrow-Debreu equilibria are much easier to calculate than incomplete markets equilibria in stationary economies—a topic to which we now turn.

Part II: Stationary Equilibria

Understanding when equilibria of the economies we have studied above exist is a first important step: studying this problem has yielded insights into the properties of equilibria and the restrictions that must be placed on the way agents trade. To go further we need to examine if the equilibrium outcomes predicted by these models correspond—or in what way they fail to correspond—with observed sequences of trades and security prices on financial markets: in short we must try to take the model to the data and examine the weaknesses and strengths of the predictions they make. To do this, we must be able to *calculate* the equilibria once the exogenous shock process and agents' endowments and preferences have been specified. Research so far has focused on the simplest class of cases where the calculation can be made.

At the beginning of the story is a specification of the exogenous shock process, and the simplest class of stochastic processes are the time-homogeneous Markov processes. Suppose the event tree $\mathbb{D}$ is generated by such a process. At each date t a shock $s_t \in S$ occurs, where S is a finite set—to keep notation simple we let the same symbol S denote both the set and the number of its elements. A node ξ at date t is then just a history of shocks, $\xi = (s_0, s_1, \ldots, s_t)$, and a successor $\xi' \in \xi^+$ can be written as $\xi' = (s_0, s_1, \ldots, s_t, s_{t+1})$, with $s_{t+1} \in S$. The exogenous shock process is *Markov* if the probability of node ξ' conditional of having reached node ξ only depends on s_t and s_{t+1}. Let $M = [M_{s,s'}, s, s' \in S]$ denote the $S \times S$ *transition matrix* where $M_{s,s'}$ denotes the probability of shock s' at date $t+1$ given shock s at date t. Thus we are assuming that the exogenous shock process satisfies

$$Prob\,[\,\xi' \,|\, \xi\,] = Prob\,[\,s_{t+1}|s_t\,] = M_{s_t, s_{t+1}}$$

If the economy begins with the shock s_0 at date 0, then the probability of node $\xi = (s_0, s_1, \ldots, s_t)$ is given by $\rho(\xi) = M_{s_0,s_1} \times \ldots \times M_{s_{t-1},s_t}$.

A Markov structure is carried over to the characteristics of the overall economy by assuming that agents' endowments and the payoffs of the securities at a node ξ only depend on the current exogenous shock s_t and that agents' preferences are additively separable

$$u^i(x^i) = \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} \rho(\xi) v^i(x^i(\xi), s_{t(\xi)}), \quad i \in I \tag{13}$$

where $t(\xi)$ and $s_{t(\xi)})$ denote the date and current shock at node ξ. In general it is assumed that all agents have the same discount factor δ. This assumption is needed to obtain a stationary equilibrium in the simple case of complete markets, since it is well known that with complete markets impatient agents are progressively driven out of the market by more patient agents.

The simplest form that an equilibrium can have for such an economy is that the current consumption, portfolios and prices only depend on the current shock s_t. For then an equilibrium can be summarized by a map from the state space S to the appropriate Euclidean space $\mathbb{R}^n$ which associate to each $s \in S$ the current equilibrium $(x(s), z(s), p(s), q(s))$. Such an equilibrium exists for the Arrow-Debreu market structure, and thus for an economy with a sequentially complete market structure (at each node there are S securities with linearly independent payoffs). This property is a consequence of the Pareto optimality of an AD equilibrium. For suppose that ξ and ξ' are two nodes which have the same current shock $s_{t(\xi)}) = s_{t(\xi')}$ but for which agents' consumption streams are not the same $x(\xi) \neq x(\xi')$. Since agents' endowments and security payoffs are the same at the two nodes, the aggregate resources are the same. Thus the allocation which offers the agents the consumption

$$\tilde{x}(\xi) = \tilde{x}(\xi') = \frac{\delta^{t(\xi)}\rho(\xi)x^i(\xi) + \delta^{t(\xi')}\rho(\xi')x^i(\xi')}{\delta^{t(\xi)}\rho(\xi) + \delta^{t(\xi')}\rho(\xi')}$$

and is otherwise the same as before, is feasible and strictly preferred if agents' utility functions v^i are strictly concave, i.e. if agents are risk averse. When markets are complete, agents can trade away their individual risks and their share of the aggregate risk depends only of the current exogenous shock.

When markets are incomplete it can be shown by a simple argument that generically this property no longer holds (see Kubler and Schmedders (2003)). Intuitively when agents can no longer fully insure all their individual risks, the history of the sequence of shocks influences the wealth that an agent brings to a node: two paths (nodes) in the event tree—one with a history of "bad" shocks, the other with an history of "good" shocks for agent i— both with the same current shock, will imply, if there is not perfect insurance, that the consumption and portfolio of agent i are different, since the wealth accumulated on the favorable path will in all likelihood be greater than that on the unfavorable path. Thus with incomplete markets, history matters.

However there is still a way of expressing an equilibrium as function of a "state" following a Markov process if endogenous state variables which summarize the past are added to the exogenous shock which affects the current endowments and payoffs. The focus of the recent literature is on finding the minimum number of endogenous variables which need to be introduced to obtain a Markov representation of an equilibrium.

Finding endogenous variables which permit a Markovian representation of the solution of a maximum problem formulated via dynamic programming has been extensively studied in macroeconomics. However finding endogenous variables which permit a Markovian representation of equilibria is the subject of much more recent research. The first formalization was given in the paper of Duffie, Geanakoplos, Mas-Colell and McLennan (1994, Chapter 7, DGMM for short): the paper addresses two issues. First it gives an abstract definition of a Markov equilibrium for an economy with Markov characteristics and shocks: this framework is sufficiently general to cover a wide class of equilibrium problems, not only for infinite-lived agents but also for overlapping generations models. Second it provides a framework for establishing the existence of a generalized steady state for the induced dynamical system, by showing that the Markov equilibrium process has an invariant ergodic measure on the generalized state space. The subsequent literature, as represented by the paper of Kubler and Schmedders (2004) in Chapter 8, has focused on the first issue—in particular on reducing the number of endogenous variables required to express the equilibrium in Markovian form. The literature so far has not focused on the equally important problem of finding minimal conditions under which a Markovian equilibrium has an invariant ergodic measure.

In this Introduction we present a simplified version of the DGMM approach which focuses on the first issue, namely the definition and existence of a Markov equilibrium for the class of economies with infinite-lived agents studied in this volume. This may help the reader to understand the more abstract framework of DGMM, which is needed to study the second issue, namely the existence of an invariant ergodic measure.

Consider an economy in which the event-tree $\mathbb{D}$ is generated by an exogenous Markov process on the finite set S introduced above–namely a Markov chain with transition matrix M. We focus on the simplest class of finance (one-good) economies in which agents' endowments and the payoffs of the J securities are such that

$$\omega^i(\xi) = \omega^i(s_t), \qquad V^j(\xi) = V^j(s_t), \quad \forall\ \xi = (s_0, s_1, \dots, s_t) \in \mathbb{D}, \quad \forall\ i \in I, \quad \forall j \in J$$

i.e. the agents' endowments and the security payoffs at a node only depend on the current shock. Financial markets are incomplete. For simplicity we assume that all securities are long lived, in positive supply, and that agents cannot short sell. Agents' preferences are assumed to have the additive separable form (13).

To show that there exists an equilibrium with a Markov representation, DGMM do not seek a state space with a "minimal" number of endogenous variables. Instead they define a "state" to be a complete description of the current exogenous shock and all the endogenous equilibrium variables at a node, namely the agents' beginning-of-period portfolios z^-, inherited from the past, their end-of-period portfolios z, the security prices q and the resulting consumption x

$$\sigma = (s, z^-, z, q, x) \in S \times \mathbb{R}_+^{JI} \times \mathbb{R}_+^{JI} \times \mathbb{R}_+^{J} \times \mathbb{R}_+^{I}$$

The states of interest are those of the set Σ which incorporates the feasibility conditions for consumption and for portfolios

$$\Sigma = \left\{ \sigma \in S \times \mathbb{R}^n \;\middle|\; \begin{array}{l} x^i = \omega^i(s) + (V(s) + q) z^{i-} - q z^i, \ i \in I \\ \sum_{i \in I} z^{i-} = \mathbf{1}, \quad \sum_{i \in I} z^i = \mathbf{1} \end{array} \right\}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^J$. In order that a state be part of an equilibrium, each agent's portfolio z^i must be optimal given the consumption that it implies for the next period. The DGMM approach is based on considering the correspondence G which associates with σ all the states that can follow and justify that agents chose the portfolios z at the prices q implied by σ. To express this condition, a notation similar to that used for event trees is useful. Let σ^+ denote a possible continuation (set of successors) of σ, each continuation being an S-vector

$$\sigma^+ = \Big((s', z^{-\prime}, z', q', x') \Big)_{s' \in S}$$

For any successor $\sigma' \in \Sigma^+$ it is convenient to write

$$(s', z^{-\prime}, z', q', x') = (s(\sigma'), z^-(\sigma'), z(\sigma'), q(\sigma'), x(\sigma'))$$

The correspondence $G : \Sigma \to \Sigma^S$ is the set of all possible continuations σ^+ of σ for which the portfolios $z(\sigma)$ are optimal

$$G(\sigma) = \left\{ \sigma^+ \in \Sigma^S \;\middle|\; \begin{array}{l} z^-(\sigma') = z(\sigma), \quad \sigma' \in \sigma^+ \\ \Big[v^{i\prime}(x^i(\sigma)) q_j(\sigma) - \sum_{\sigma' \in \sigma^+} M_{s(\sigma), s(\sigma')} v^{i\prime}(x^i(\sigma'))(V^j(s(\sigma')) + q_j(\sigma')) \Big] \le 0 \\ [\quad] \, z_j^i(\sigma) = 0, \quad j \in J, \quad i \in I \end{array} \right\}$$

where $[\quad] \, z_j^i(\sigma)$ expresses the complementary slackness condition for the no-short sales constraint on security j, $[\quad]$ denoting the expression between brackets on the preceding line.

The first condition in the definition of G expresses a compatibility condition between σ and each successor $\sigma' \in \sigma^+$: the beginning-of-period portfolio $z^-(\sigma')$ at a successor node must be the end-of-period portfolio of σ. The second and third conditions express the FOCs for an optimal portfolio for each agent, given the consumption and payoffs of securities across the nodes $\sigma' \in \sigma^+$. DGMM add a lower bound on the consumption of each agent, but we will not enter into technical boundary conditions here.

A set $J \subset \Sigma$ is said to be *self-justified* for G if for each $\sigma \in J$, $G(\sigma) \cap J^S \neq \emptyset$. A *Markov equilibrium* is a pair (J, F) consisting of a compact self-justified set J for G and a map $F : J \to J^S$ which is a selection of the correspondence G, i.e. is such that $F(\sigma) \in G(\sigma)$ for all $\sigma \in J$. The probability of a transition from a state $\sigma \in J$ to the state $F(\sigma)_{s'}$ is $M_{s(\sigma),s'}$. Using the map F, an equilibrium can be written heuristically as the solution of a first-order stochastic difference equation

$$\sigma_t^+ = F(\sigma_t) \tag{14}$$

More precisely, if $\sigma_0 \in J$ is a state at date 0 then $F_{s_t} \circ \ldots \circ F_{s_1}(\sigma_0)$ for all $\xi = (s_0, s_1, \ldots, s_t) \in \mathbb{D}$ is an infinite horizon equilibrium.

DGMM give a simple and ingenious argument to show that a Markov equilibrium exists as soon as' for every finite time horizon T, the economy truncated at T has a finite-horizon equilibrium in which all the equilibrium variables lie in a compact set, which is independent of T. Proving this property is the first step to proving existence of a (non-necessarily Markov) infinite horizon equilibrium (see Chapters 1,2,3,5).

Intuitively the argument amounts to looking at the date 0 component of all infinite horizon equilibria with all possible initial conditions (s_0, z_0^-), calling this set J. The mapping from J to J^S is then obtained by associating with any date 0 equilibrium the value of the equilibrium variables at date 1 for all exogenous shocks $s' \in S$. More formally the set J is defined by induction. Let $K = S \times B$ be a compact set where B is a compact set of $\mathbb{R}^{IJ} \times \mathbb{R}^{IJ} \times \mathbb{R}^J \times \mathbb{R}^I$ in which all the variables $(z(\xi)^-, z(\xi), q(\xi), x(\xi))$ of an economy truncated at T, for any T must lie. Define $C_0 = K^S$ and define

$$C_1 = \{\sigma_0 \in K \mid G(\sigma_0) \cap C_0 \neq \emptyset \}$$

C_1 is not empty since it contains the date 0 variables of 2-period equilibria. Define recursively

$$C_2 = \{\sigma_0 \in K \mid G(\sigma_0) \cap C_1 \neq \emptyset \}, \ \ldots, \ C_T = \{\sigma_0 \in K \mid G(\sigma_0) \cap C_{T-1} \neq \emptyset \} \tag{15}$$

C_T contains the date 0 variables of the T-period equilibria. Note that $C_T \subset C_{T-1} \subset \ldots \subset C_0$.

Then define

$$J = \bigcap_{T \geq 0} \overline{C}_T \tag{16}$$

where $\overline{C}_T$ denotes the closure of the set C_T. J is non-empty as the intersection of a decreasing sequence of non-empty compact sets. Let $\sigma \in J$ and $(\sigma_T)_{T \geq 0}$ be an infinite sequence with $\sigma_T \in C_T$ such that $\sigma_T \to \sigma$. Let $F(\sigma_T)$ be an element of $G(\sigma_T) \cap C_{T-1}$; there is a convergent sequence $(F_{T_n}(\sigma_{T_n}))$ and let $F(\sigma)$ be defined by $F(\sigma) = \lim F_{T_n}(\sigma_{T_n})$. Since G (being defined by continuous functions and weak inequalities) has a closed graph, $F(\sigma) \in G(\sigma) \cap J$. Repeating the argument for each $\sigma \in J$ leads to a map $F : J \to J^S$ which is a selection of the correspondence G, so that (J, F) is a Markov equilibrium.

For an economy with given characteristics, finding the self justified set J and the map F which gives the representation (14) is tantamount to calculating all infinite horizon equilibria for all possible initial conditions: hardly a progress from a computational point of view. In practice when researchers have sought to calculate equilibria of infinite horizon economies with incomplete markets they have used methods inspired by the approach of dynamic programming. Finding the solution of a dynamic programming problem consists in finding a policy function which expresses the optimal current decision as a function of the exogenous shock and appropriate endogenous variables—generally called *state variables*—which are sufficient to summarize the past. The solution of the intertemporal problem is then determined by the policy function and the law of motion of the state variables. The paper by Kubler and Schmedders (2004, Chapter 8) shows how an equilibrium version of the policy function approach can be embedded into the DGMM framework, so that the existence of an equilibrium "policy function" representation can be addressed.

In the framework of the model studied above, a policy function is a map $\phi : S \times \mathbb{R}^{IJ}_+ \to \mathbb{R}^{IJ}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^I_+$ which associates with a value of the state variables (s, z^-), the endogenous variables (z, q, x) of the current equilibrium. In order that ϕ represents a Markov equilibrium it must satisfy

(i) $\text{graph}(\phi) \subset \Sigma$

(ii) $F(\sigma) \doteq (s', z(\sigma), \phi(s', z(\sigma))_{s' \in S} \in G(\sigma), \quad \forall\, \sigma = (s, z^-, \phi(s, z^-))$

The idea is that, given the current shock s which determines the current endowments and security payoffs, and z^- which determines the distribution of wealth inherited from past transactions, each agent has a rule for choosing the new portfolio which optimally prepares him for future contingencies, so that security prices which clear the markets depend on the same current and wealth variables. Condition (i) requires that the domain of ϕ is restricted to the beginning-of-period portfolios satisfying $\sum_{i \in I} z^{i\,-} = \mathbf{1}$, and that the values (z, q, x) of ϕ satisfy the market clearing conditions $\sum_{i \in I} z^i = \mathbf{1}$ and the budget constraints $x^i = \omega^i(s) + (V(s) + q)z^{i\,-} - qz^i$. Condition (ii) expresses the law of motion of the state variable $z^-(\sigma') = z(\sigma)$ for all $\sigma' = (s', F_{s'})(\sigma)$, and the property that if the next period choices are given by the function ϕ, then the current portfolio choices in $\phi(\sigma, z^-)$ are optimal.

One way of calculating an approximate function $\phi(\sigma, z^-)$ satisfying (i) and (ii) is to imitate the procedure of "value function iteration" in dynamic programming. The candidates functions ϕ are chosen from a family of functions characterized by finitely many parameters, for example piecewise linear or cubic spline functions on a grid for the portfolios. Starting with a function $\phi^{(0)}$ in the family with graph$(\phi^{(0)}) \in \Sigma$, one looks for a function $\phi^{(1)}$ satisfying

(i′) graph$(\phi^{(1)}) \subset \Sigma$

(ii') $\left(s', z^{(1)}, \phi^{(0)}(s', z^{(1)})\right) \in G\left(s, z^-, \phi^{(1)}(s, z^-)\right)$

where $z^{(1)}$ is the portfolio component of $\phi^{(1)}(s, z^-)$. Finding $\phi^{(1)}$ amounts to solving a finite number of times (S times the number of elements of the grid) a finite system of equations describing a two period equilibrium such that the portfolio decisions are optimal given that the way an agent's portfolio decision influences consumption tomorrow is given by the function $\phi^{(0)}$. Repetition of the procedure gives rise to a sequence of functions $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(n)}$ and, provided the procedure converges, it terminates with a function $\hat{\phi}$ which is an approximate solution of (i) and (ii). In Chapter 8, KS give the details of the calculation. They use wealth as a state variable rather than portfolios since it decreases the dimension of the state variables, but the principle is the same.

At the present time no general conditions are known on the characteristics of an economy which guarantee that there exists a Markov equilibrium which can be described by a policy function ϕ on the portfolio or wealth space satisfying (i) and (ii). KS show in Chapter 8 that the existence of a Markov equilibrium (J, F) in the sense of DGMM implies the existence of a correspondence $\Phi : S \times \mathbb{R}_+^{IJ} \to \mathbb{R}_+^{IJ} \times \mathbb{R}_+^J \times \mathbb{R}_+^I$ satisfying

(i″) graph$(\Phi) \subset \Sigma$

(ii″) $F(\sigma) \in (s', z(\sigma), \Phi(s', z(\sigma))_{s' \in S}\,, \quad \forall\, \sigma \in \text{graph}(\Phi)$

But the existence of a function ϕ satisfying these conditions is not guaranteed. The problem arises if a DGMM equilibrium is such that there exist two equilibrium states $\sigma = (s, z^-, z, q, x)$ and $\tilde{\sigma} = (\tilde{s}, \tilde{z}^-, \tilde{z}, \tilde{q}, \tilde{x})$ with $z = \tilde{z}$ and $F(\sigma) \neq F(\tilde{\sigma})$. Since $F(\sigma) \neq F(\tilde{\sigma})$, there exists a state s' such that $F(\sigma)_{s'} = (s', z, z', q', x') \neq F(\tilde{\sigma})_{s'} = (s', z, \tilde{z}', \tilde{q}', \tilde{x}')$. Then there exists two equilibrium states with the same "state variables" (s', z) and different equilibrium variables. In this case $\Phi(s', z)$ must contain at least two elements.

The problem is linked to the multiplicity of equilibria. From the construction of a Markov equilibrium the situation just described will only arise if there are two infinite horizon equilibria with the same initial conditions (s', z). Kubler-Schmedders (2002) exhibits a (rather involved) example which has the property that all Markov equilibria are such that there exist two states σ and $\tilde{\sigma}$ with $z(\sigma) = z(\tilde{\sigma})$ and $F(\sigma) \neq F(\tilde{\sigma})$. The same problem exists in OLG Markov economies with several goods and/or agents living for more than two periods. In the OLG setting examples are easier to construct because of the finite life-span of the agents (see Kubler-Polemarchakis (2004)). Studying whether the property that there exist σ and $\tilde{\sigma}$ with $z(\sigma) = z(\tilde{\sigma})$ and $F(\sigma) \neq F(\tilde{\sigma})$ in all DGMM equilibria is robust or disappears with a small perturbation of the economy is the object of current research.

Part III: Effect of Incomplete Markets on Allocations, Welfare and Prices

In the early 1990's, at the same time that the abstract properties of the infinite-horizon version of the GEI model were being developed, a parallel literature in macroeconomics began to explore the properties of the equilibria of such models using the approach of calibration, with a focus on the interest rates, equity prices and capital accumulation predicted by the model. In order to use the tools and methods of dynamic programming to calculate equilibria, the economic models were kept as close as possible to the representative-agent model. However for the strict representative agent model, which is simply a special case of the Arrow-Debreu model with complete markets, Mehra-Prescott (1985) had shown that the interest rates predicted by a calibrated version of the model are typically too high, and the equity premium too low. A motivating idea in several of the papers of Part III is that moving away from the Arrow-Debreu model to a model with incomplete markets could serve to improve the predictive power of the model: if risk markets are incomplete, agents may be induced to save more for the rainy days, thereby lowering interest rates and, being exposed to a larger variability of consumption, may have a decreased willingness to bear risk, thereby requiring a larger equity premium.

In most of the papers of this Part agents are assumed to be ex-ante identical, with preferences of the expected utility, constant-relative-risk-aversion (CRRA) form typical of macroeconomics, with fluctuating endowments. All the authors make a distinction between idiosyncratic risks for the agents, which are assumed to be non-insurable, and aggregate risks which are traded on a stock market. In the simplest models (Huggett, (1993, Chapter 9, Aiyagari (1994, Chapter 10)) aggregate risks are neglected and the aggregate consumption is constant. The underlying theme of this literature is that the representative-agent model underestimates the impact of risks in the economy by assuming (implicitly) that individual risks are optimally shared and that the only risks that agents bear are the aggregate risks which, when calibrated, do not have a large variance. In the real world agents are subjected to a wide variety of risks, of which only a small proportion (e.g fire and theft) are insurable. The major risks arising from shocks to labor income are essentially uninsurable due to moral hazard.

However it turns out that a “simple” formalization of this idea in the setting of a dynamic intertemporal model does not give the result that intuition—and the two-period model—might suggest. Replacing the single representative agent by a continuum of agents with uninsured idiosyncratic risks is not sufficient to obtain an equilibrium that differs from the complete markets equilibrium, as long as the idiosyncratic risks are stationary and there is one market on which agents can trade, which permits them to transfer purchasing power from one period to the next without (too much) friction. For in a dynamic setting with a long horizon and unencumbered access to a single asset, a new technique enters the kit bag of an agent ,which is one of the most powerful instruments for taming the random fluctuations of an agent’s income stream. For a stationary random income stream has a long-run average and applying “carry-over” strategies—buying (saving) when the income realization is above this average and selling (dissaving) when it is below average—essentially permits the risky income stream to be transformed into a constant, non-random consumption stream. The papers on equilibrium with idiosyncratic income risks in Part III essentially rediscover the power of “carry-over” strategies, a tool employed by societies for many centuries for smoothing random harvests of a storable commodity. To permit a perfect transformation of the random income stream into the riskless long-run average, no ‘loss’ must be incurred in transferring income for one period to the next, so that the interest rate must be zero. However the papers in Part III show that even with a positive interest rate, carry-over strategies work remarkably well for achieving consumption smoothing.

One of the first steps in the macro literature towards the analysis of an equilibrium model with idiosyncratic risks may be found in the paper of Huggett (1993, Chapter 9). Huggett considers an economy with a continuum of agents with the same CRRA expected discounted utility where agents have endowments processes which are independent (across agents) and identically distributed, each endowment ω^i following a two-state Markov chain with outcomes ω_H or ω_L. At each date-event a one-period riskless bond is traded and agents can lend as much as they want at the current interest rate but are limited in the amount they can borrow by a credit limit which is a parameter of the model, and which can be made more or less binding. The analytical framework adopted by Huggett is that of dynamic programming: a stationary portfolio policy $z(\omega, z^-; q)$ is sought for an agent with current endowment $\omega \in \{\omega_H, \omega_L\}$, inheriting a bond holding z^- from the previous period and facing the fixed bond price q for ever. Showing that such a policy exists is in essence a standard argument of dynamic programming. For fixed q, Huggett then looks for the stationary distribution $\Psi(\omega, z^-; q)$ of the agents across endowment and asset holdings associated to the policy $z(\omega, z^-; q)$, using a result of Hopenhayn-Prescott (1992) on the existence of an invariant distribution of monotone Markov processes to show that such a distribution exists. A stationary equilibrium is then a triple $(z(\omega, z^-; q^*), \Psi(\omega, z^-; q^*), q^*)$ such that $z(\omega, z^-; q^*)$ is the optimal policy given q^*, $\Psi(\omega, z^-; q^*)$ is the associated invariant distribution, and the bond market clears i.e. $\int z(\omega, z^-; q^*) d\Psi(\omega, z^-; q^*) = 0$. While the existence of an optimal policy function and an invariant distribution are established analytically, the existence of a market clearing price is not proved, instead (approximate) equilibrium prices are found in calibrations. The existence of a market clearing price can probably be deduced from the boundary behavior of the policy function $z(\omega, z^-; q)$: if q is sufficiently high all agents tend to be borrowers and if q is sufficiently low all agents tend to be savers, so that a bond price q^* which clears the market (at all time) exists.

In the complete markets version of the model, agents' consumption streams are constant and the equilibrium interest rate equals the pure rate of impatience $1/\delta - 1$ of any agent. Huggett showed through calibration that in a stationary equilibrium the interest rate $r^* = 1/q^* - 1$ is less than the pure rate of impatience $1/\delta - 1$, or equivalently $q^* > \delta$. This can in fact be established analytically in two ways—one is to draw on the known properties of the income fluctuation problem which has been extensively studied in an earlier literature (Yaari (1976), Schechtman (1976), Schechtman-Escudero (1977), Bewley (1977)). Using the martingale convergence theorem it can be shown that if the interest rate is greater or equal to the rate of impatience, and income is fluctuating, then the asset holding of each agent grows without bound as time goes to infinity (see Chamberlain-Wilson (2000)). Since this is incompatible with equilibrium, the equilibrium interest rate must be less than the pure rate of impatience. The second, and simpler method of proving this property is to assume convexity of the agents' marginal utilities, an assumption made by Levine and Zame in Chapter 12, which we will discuss shortly.

Huggett found however that for "reasonable" calibrations of agents' preferences and idiosyncratic risks, the difference in interest rates between the representative agent economy and the economy with uninsurable risks in which agents have non-binding borrowing constraints is small. In retrospect this is not surprising since with non-binding borrowing constraints the generalized carry-over strategies of bond holdings, saving or decreasing debt when the income realization is above average and disaving or borrowing when the income realization is below average allow agents to achieve an essentially constant consumption over time. Huggett showed that when agents' carry-over strategies are hampered by tight borrowing constraints, then the interest rate decreases significantly. With tight borrowing constraints agents tend to rely more on saving to avoid low consumption when the income draw is unfavorable, while the demand for borrowing is lower due to the constraints, so that the interest rate must decrease to discourage agents from saving more than the market can absorb.

Exploring the consequences of uninsured idiosyncratic risks for the standard growth model was the next natural step. Such analysis was performed by Aiyagari (1994, Chapter 10) who considers a model with basically the same structure as Huggett—a continuum of agents with the same CRRA discounted expected utility and idiosyncratic shocks to their labor income, except that production is explicitly modeled. Each agent has an inelastic supply of labor which is subject to idiosyncratic shocks following a Markov process: in the simplest case, at date t an agent can have either ℓ_H if the shock is favorable , or ℓ_L if the shock is unfavorable. In the calibration the Markov chain is chosen so as to approximate a first-order auto regressive process with given parameters, but in all cases, invoking a Law of Large Numbers, the total supply of labor L is constant. Agents hold a durable good, called capital, which depreciates at the rate μ: as in the standard growth model one unit of capital can be transformed into one unit of the consumption good and conversely. Agents sell their labor and rent their capital to the production sector operating with a constant-returns-to-scale production function $F(K, L)$. Aiyagari looks for a stationary equilibrium of the economy, namely a constant interest rate r, which yields a demand for capital by the production sector $K(r)$ (defined by the FOC $r = F_K(K(r), L) - \mu$), a wage rate $w(r) = F_L(K(r), L)$, a policy function $\tilde{k}(k^-, \ell; r, w(r))$ for an agent with current labor endowment ℓ and inherited capital k^-, and an associated invariant distribution $\Psi(k-, \ell; r, w(r))$ of agents across asset holding/labor endowment (k, ℓ) such that

$$\int \tilde{k}(k^-, \ell; r, w(r)) d\Psi(k^-, \ell; r, w(r)) = K(r)$$

i.e the supply of capital $E\tilde{k}(r)$ by agents equals the demand $K(r)$ of capital by firms. It can be deduced from Chamberlain-Wilson (2000) that the supply of capital $E\tilde{k}(r)$ tends to infinity when r tends to the pure rate of impatience $1/\delta - 1$, while the demand for capital is a decreasing function which goes to infinity when r tends to μ and tends to zero when r tends to infinity. Thus the supply and demand curves intersect and there is a stationary equilibrium interest rate r^*. Since the supply of capital tends to infinity when r tends to the pure rate of impatience, which is the representative-agent equilibrium interest rate, the presence of uninsured labor risks lowers the interest rate and increases the steady-state capital.

Aiyagari calibrates the model with CRRA utility with a coefficient of relative risk aversion γ, a first-order auto regressive process for the log of the labor endowment process with serial correlation ρ and coefficient of variation σ, and studies the stationary equilibrium interest rate as a function of these parameters (γ, ρ, σ). As expected the interest rate decreases when risk aversion γ and exposure to risk σ increases since an increase in these parameters increases the demand for precautionary savings. Perhaps Aiyagari's most interesting observation is that with reasonable but relatively high risk aversion and exposure to risk, increasing the coefficient of serial correlation ρ dramatically decreases the interest rate: in this case agents increase their buffer stock of capital for insuring against adverse labor outcomes since an adverse outcome is likely to be followed by a series of adverse shocks. This increase in agents' buffer stocks raises the capital stock and decreases the interest rate.

An interesting aspect of these highly stylized models of incomplete markets for idiosyncratic risks is that, although agents are identical ex-ante, they will find themselves after the passage of time in different “quintiles” of the wealth distribution: for an agent who has been exposed to a relatively high proportion of adverse shocks will find himself in the lowest quintile, while an agent who has found fortune’s favors on his side will end up in the highest quintile. The model thus suggests comparing the wealth distribution generated in a stationary equilibrium with the actual distribution of wealth, say in the US, at some moment of time. Aiyagari examines the Lorenz curve and associated Gini coefficient generated by calibrated versions of the model and finds a much lower Gini coefficient (of the order of 0.1) than the US Gini coefficient (of the order of 0.8): the concentration of wealth at the upper end is much greater in the US than that delivered by this model.

The models of Huggett and Aiyagari deliver a stationary equilibrium with a constant interest rate. This property hinges on the absence of aggregate risk. Krusell and Smith (1998, Chapter 11) study essentially the same model as Aiyagari adding a productivity shock to the aggregate production function which is $\epsilon_H F(K, L)$ if the shock is favorable and $\epsilon_L F(K, L)$ if the shock in unfavorable. Adding aggregate risk in this way considerably complicates the concept of equilibrium and its calculation. For the models of Huggett and Aiyagari can be solved in three steps using well-known results and techniques of dynamic programming: the first step consists in deriving the optimal policy function for an agent facing a constant interest rate forever; the second step finds the associated stationary distribution of agents across shocks-asset holdings and, in the final step, the equilibrium problem consists in adjusting the interest rate to clear the financial market. As soon as there are aggregate risks the interest rate and the wealth distribution vary with the aggregate shock: depending on whether the return to capital is high or low the same distribution of wealth and aggregate capital at date t will result in a different distribution at date $t+1$. The natural concept of equilibrium now becomes that of a recursive equilibrium studied in Part II, with state variables (Ψ, ϵ), where Ψ is the distribution of agents across capital and labor shocks, and ϵ is the productivity shock. Instead of a stationary distribution and a constant interest rate, the equilibrium becomes a law of motion $\Psi' = H(\Psi, \epsilon, \epsilon')$ and a policy function for the choice of capital by a typical agent $\tilde{k}(k^-, \ell : \Psi, \epsilon)$ as a function of the inherited capital, the labor shock, and the aggregate state variable (Ψ, ϵ), such that

(i) $\tilde{k}$ is optimal given the law of motion H and the Markov structure for the individual and aggregate risks

(ii) H is the law of motion generated by $\tilde{k}$ and the Markov structure.

The current prices—the wage and the interest rate—are the marginal product of labor and capital and thus only depend on the aggregate capital K, which is the mean of Ψ with respect to k. The exact form of the distribution Ψ only matters for the law of motion of Ψ. If poor agents have a propensity to save which is different from that of rich agents, two distributions with the same mean at t but different Gini coefficients can generate different wealth distributions at date $t+1$ with different means, and hence different wages and interest rates. This heuristic argument suggests conditions under which it may be reasonable to simplify the computation of equilibrium by replacing the law of motion of Ψ by a law of motion for the mean of Ψ: if agents' preferences are such that the propensity to save at different wealth levels is approximately the same, then replacing the distribution by its mean may yield an approximate equilibrium which is "not far" from a true equilibrium—although it is not clear how to evaluate the magnitude of the error. Krusell and Smith adopt the procedure of replacing the wealth distribution by its mean to find the equilibrium prices, and argue that the method provides a good approximation because, for a given labor endowment shock, the solution to the dynamic programming choice problem of a typical agent exhibits an almost constant propensity to save out of inherited wealth, except for the lowest wealth levels, where agents essentially do not invest in capital, and thus do not influence the formation of capital in the next period. Thus Krusell and Smith replace the state variable (Ψ, ϵ) by the state variable (m, ϵ), where m is the mean of Ψ with respect to k, and replace the law of motion of Ψ by a law of motion $m' = h(m, \epsilon, \epsilon')$. Since the scalar m replaces the entire distribution function Ψ, this approximate equilibrium is much simpler to calculate. The approximation can be improved by replacing m by a vector of central moments of Ψ with respect to k, evaluating a functional form for the law of motion of the moments. The claim is that when higher order central moments are included, the equilibrium—consumption, capital and interest rate processes—des- not change significantly.

While adding aggregate shocks to production adds realism to the model, it does not solve the weakness of Aiyagari's model in mimicking the observed wealth distribution. One solution is to make a radical change to the labor income process, using a finite Markov process with very substantial differences in labor endowments and a very high degree of permanence which locks the agents into a given income level. This is the procedure adopted in a recent paper of Davila-Hong-Krusell-Rios-Rull (2007) who show that the resulting equilibrium gives a Gini coefficient for the wealth distribution which conforms much more closely with that of the US economy. Another approach is to view the differences in wealth as the result of differences in the preferences of the agents: Krusell and Smith explore a version of their model in which agents have random time-discount factors δ_t^i, the randomness of δ_t^i permitting them to avoid the well-known property that agents with low discount factors will be driven out of the market. The randomness of δ_t^i allows them to model crudely in an infinite-lived agent model the OLG-like idea that preferences may vary randomly across generations of the same dynasty. They show that introducing such a randomness in discount factors of families over time leads to a much more realistic Lorenz curve and Gini coefficient for the wealth distribution in the calibrated model. This appears to be an interesting and fruitful way of exploring heterogeneity in the equilibrium model while retaining simplicity in calibration.

A common element in the calibrated equilibria studied in Chapters 9-11 is the property that, with access to a single financial instrument like a bond (Huggett) or a capital asset (Aiyagari, Krusell-Smith), agents can transfer income from 'good' years to 'bad' years thereby obtaining a relatively stable consumption stream, despite considerable variability in the agents' original income streams. To work well the carry-over strategies must be unhampered—there must not be borrowing constraints which are too tight nor must the income shocks be subject to too much serial correlation (permanence). While carry-over strategies for durable goods and borrowing/lending have been used for many centuries as instruments for smoothing consumption over time, the formal mathematical analysis of the conditions under which such strategies work is of much more recent origin. The first theoretical papers seem to be those of Yaari (1976) which studies consumption smoothing through borrowing and lending, and Schechtman (1976) which studies consumption smoothing through carry-over—i.e. by holding a long position in a durable asset. Both papers study the case where an agent has a zero pure rate of impatience (i.e. $\delta = 1$). Yaari assumes that the interest rate is zero and Schechtman makes the analogous assumption that the durable asset can be carried over without loss from one period to the next. Yaari shows that if the agent is subject to independent and identically distributed income shocks, i.e. the random variables $(\omega_t)_{t\geq 0}$ are i.i.d. then, as the agent's horizon T tends to infinity, the consumption x_t^T at any given time t in the T-period economy converges to $E(\omega_t)$ with probability one. Since achieving complete consumption smoothing by carry-over of a durable asset is harder—for an agent needs to have time to build up a sufficient buffer stock—Schechtman established the weaker result that as t and T tend to infinity , x_t^T converges to $E(\omega_t)$ with probability one. These results show that an agent can essentially smooth disposable income—complete consumption smoothing with borrowing and lending, asymptotic consumption smoothing with carry-over— when the cost of transferring income is zero, i.e. agents are not impatient and the rate of interest (rate of return) is zero.

These results shown by Yaari and Schechtman are properties of an individual agent's maximum problem, and are not results shown to hold in equilibrium. Indeed Schechtman's result cannot hold in equilibrium if there is a bound on the aggregate supply of the asset that agents carry over: for every agent is trying to self-insure by building up an arbitrarily large buffer stock, which is collectively impossible if the supply of the asset is limited. Yaari's borrowing and lending framework is however more flexible for, when it is transposed to an equilibrium setting in which there is a continuum of agents with zero pure rate of impatience and i.i.d. endowments (independent across agents and time), then the result of perfect consumption smoothing through borrowing and lending continues to hold. The equilibrium interest rate is zero because there is no aggregate risk and no impatience, so that agents can combine self insurance with collective insurance to obtain riskless consumption streams. This result on perfect consumption smoothing does not hold when agents are impatient, for then the interest rate is positive and this introduces a 'cost' to transferring income over time by borrowing and lending. Levine and Zame (2002, Chapter 12) (LZ for short) show however that perfect consumption smoothing can be obtained as the limit of equilibria of economies without aggregate risk when the agents' pure rate of impatience tends to zero.

Instead of considering an economy with a continuum of agents Levine and Zame consider an economy with a finite number of agents defined over an event tree $\mathbb{D}$, in which the sum of the agents' endowments is assumed to be constant at all date-events so that there is no aggregate risk: $\sum_{i\in I}\omega^i(\xi) = w$ for all nodes $\xi \iota \mathbb{D}$. Each agent maximizes an expected discounted utility $u^i_\delta(x^i) = (1-\delta)E\sum_{t=0}^{\infty}\delta^t v^i(x^i_t)$ with the same discount factor δ for all agents, δ being taken as parameter of the model. The utility function v^i is concave and the marginal utility $v^{i\prime}$ is assumed to be a convex function. In the simplest version of the model agents have access to a single security, the one-period riskless bond: adding other assets does not change the result as long as the markets remain incomplete.

In an equilibrium of the same economy with complete markets each agent's consumption is constant and the interst rate is $r^* = 1/\delta - 1$. The first step in LZ's analysis is to show that in an equilibrium with incomplete markets, the interest rate is always less than or equal to r^*. This is readily deduced from the assumption of convexity of $v^{i\prime}$which, combined with the first-order condition for an agent's bond holding gives

$$v^{i\prime}(x_t^i)q_t = \delta E v^{i\prime}(x_{t+1}^i) \geq \delta v^{i\prime}(E x_{t+1}^i) \tag{17}$$

where q_t denotes the price of the bond at date t. Since there is no aggregate risk, $\sum_{i \in I} \omega_t^i = w$ at all dates, so that there must be one agent for whom $x_t^i \geq E x_{t+1}^i$, implying $v^{i\prime}(x_t^i) \leq v^{i\prime}(E x_{t+1}^i)$. But then (17) implies that $q_t \geq \delta$, or equivalently, $r_t \leq 1/\delta - 1$. The convexity of $v^{i\prime}$ yields a precautionary demand for saving with incomplete markets since it implies the inequality

$$\frac{\delta E v^{i\prime}(x_{t+1}^i)}{v^{i\prime}(x_t^i)} \geq \frac{\delta v^{i\prime}(E x_{t+1}^i)}{v^{i\prime}(x_t^i)}$$

i.e. the value of transferring an additional unit of income from date t to date $t+1$ is greater when the agent's consumption is variable than when the consumption is constant and equal to the expected value for sure: the increased demand for saving created by the variability of consumption in an equilibrium means that the equilibrium interest rate is lower than it would be in the absence of variability. The precautionary motive is a fundamental driving force which lowers the rate of interest in an equilibrium with incomplete markets relative to an equilibrium with complete markets.

In an economy without aggregate risk in which agents have expected utility preferences which exhibits risk aversion, an allocation is Pareto optimal if and only if it gives a constant consumption stream to each agent for ever. If agents' endowment streams are driven by an underlying stationary Markov process, each agent has a long-run average endowment $\bar{\omega}^i = E(\omega_t^i)$, where the expectation is taken with respect to the invariant measure of the Markov process. Thus the allocation $x_t = (\bar{\omega}^1, \ldots, \bar{\omega}^I)$ which assigns to each agent the long-run average of his endowment for ever is Pareto optimal. LZ use the upper bound on the interest rate to prove that if the common discount factor δ of agents is close to 1, then in an equilibrium with incomplete markets $(\bar{x}, \bar{z}, \bar{q})$ an agent's lifetime expected utility $u_\delta^i(x^i)$ is close to the utility $v^i(\bar{\omega}^i)$ derived from the consumption of his long-run endowment for ever. This implies that ‘at most nodes’ the equilibrium consumption $x^i(\xi)$ is close to $\bar{\omega}^i$ and the equilibrium interest rate $\bar{r}(\xi)$ is close to zero. Thus Yaari’s result that each agent consumes his long-run average endowment is obtained as the limit of equilibria with borrowing and lending as the pure rate of time preference tends to zero, and is valid when agents have stationary Markov endowment processes, not just i.i.d. processes as in Yaari. This is indeed a beautiful result which captures in an idealized way the property repeatedly found in macro models with incomplete markets and stationary Markov endowment processes that the equilibria lie close to the complete markets Arrow-Debreu equilibria of the economy, if no additional “imperfections” are introduced. In fact, as the papers of Part III show, even if additional realistic imperfections are introduced, considerable consumption smoothing can still be achieved, either through borrowing and lending as in Huggett (Chapter 9) or by using carry-over strategies of a long lived-asset in positive supply—capital like in Aiyagari (Chapter 10) or equity as in Heaton-Lucas (Chapter 13).

To show that in equilibrium an agent’s expected utility u^i_δ converges to the utility $v^i(\bar{\omega}^i)$ of consuming the long-run average endowment for ever, Levine and Zame consider a particular feasible consumption-portfolio strategy which keeps the consumption close to the average endowment $\bar{\omega}^i$ as long as the agent’s debt is not too large. More precisely, for any small $\epsilon > 0$ which is less than the minimum endowment of any agent along the event-tree (the minimum is assumed to be strictly positive), consider $b_\epsilon = 1/2\frac{\epsilon}{r^*}$, half the value of a perpetual annuity of ϵ when the interest rate is $\rho^* = 1/\delta - 1$. As long as his debt is less than b_ϵ the agent consumes $\bar{\omega}^i - \epsilon$, borrowing when necessary, reimbursing his debt whenever possible, but never saving (i.e. consuming more than $\bar{\omega}^i - \epsilon$ when the agent has no debt and a favorable shock). If the agent has faced a series of negative shocks and hits the debt level b_ϵ, then he consumes $\omega^i(\xi) - \epsilon$, using ϵ to reimburse the debt. Since $r_t \leq r^*$ for all t, the agent reimburses his debt in finite time, the strategy does not involve any Ponzi scheme and is thus feasible. While this strategy is feasible for each agent, this is certainly not an equilibrium strategy since it is not possible in equilibrium to have only borrowers and no lenders. However this strategy provides a lower bound on the agent’s equilibrium utility level, since by optimizing an agent must do at least as well as with any feasible strategy. The proof then consists in showing, using a version of the Central Limit Theorem for Markov chains that the date at which the critical level b_ϵ is reached tends to infinity with probability 1 as $\delta \to 1$. Thus the utility for any agent yielded by this strategy tends to $v^i(\bar{\omega}^i - \epsilon)$: letting ϵ become arbitrarily small gives the desired result.

LZ also show that the result can be extended under rather restrictive assumptions to an economy with aggregate risk in which each agent's endowment is a random share of the aggregate endowment. They assume that all agents have identical CRRA expected utilities, so that the allocation in which each agent consumes his long-run average share of the aggregate endowment is Pareto optimal. They show that if agents trade the riskless bond and securities which span the aggregate risk (e.g. Arrow securities on the value of aggregate output), then the utility of each agent in equilibrium tends to the utility of consuming his long-run share of aggregate output for ever as $\delta \to 1$.

The above analysis has focused on the consumption smoothing achievable by agents facing stochastic income streams in an incomplete market equilibrium. Dual to the properties of agents' consumption streams are the properties of equilibrium prices of the assets. The analysis in Chapters 9-11 has focused on the effect of incomplete markets on the equilibrium interest rate, the main message being that in Markov economies the precautionary motive does indeed serve to lower the interest rate, but that additional imperfections or a significant degree of permanence in the shocks are needed for the effect to be substantial. A more realistic analysis of the effect of incomplete markets on the prices of securities must take into account the fact that agents trade many securities, and up to now the economic literature (in contrat to the finance literature) has focused on the 'basic' securities consisting of short-term bonds and equity on aggregate output. This is sufficient to discuss the relative prices of these two types of securities, and the "price of risk" as reflected in the equity premium required to induce agents to hold the risky security. The discussion began with the well-known paper of Mehra-Prescott (1985) which pointed out that the representative-agent model delivers an interest rate that is too high and an equity premium that is too low to match the US data. In Chapter 13 Heaton and Lucas (1996) examine whether the presence of uninsured idiosyncratic risks for the agents can generate a higher equity premium. As already in Lucas (1994), they find that even with limited access to the stock or the bond market, agents can achieve considerable consumption smoothing so that the stochastic discount factor does not fluctuate enough to generate asset prices significantly different from thise generated in the complete markets (representative-agent) equilibrium. They thus focus on the effect of transaction costs on the equilibrium prices of equity and the bond, and find that a recognition of the spread between the borrowing and lending rates combined with the presence of transaction costs on the stock market lead to an increase in the equity premium. However the order of magnitude of the transaction costs needed to generate a sizable equity premium makes it difficult to view transaction costs of trading on the capital market as a sufficient additional element to explain the equity premium.

There are in essence two ways of preventing agents from achieving consumption smoothing in equilibrium: the first is to restrict the ability of agents to apply borrowing/lending or carry-over strategies by borrowing constraints or transaction costs; the second is to move away from transient shocks of a stationary Markov process to shocks with sufficient permanence so that carry-over strategies cease to work. As we saw in Chapters 9-13, realistic applications of the first type of constraint do not significantly change the security prices from their value in the representative-agent equilibrium. The innovation in the paper of Constantinides-Duffie (1996, Chapter 14) is to introduce permanence in the shocks to agents' incomes as a potentially important explanatory factor for the equity premium. Their strategy is to exhibit an economy in which agents have permanent idiosyncratic income shocks and in which there is no consumption smoothing—there is no trade and each agent consumes his endowment income.

The economy is similar in all respects to the basic class of economies we have outlined above: a continuum of agents with expected discounted utilities in the CRRA family with the same coefficient of risk aversion α and the same discount factor δ, and a collection of securities (incomplete) which agents can trade. The difference lies in the nature of the endowment processes for the agents. Let $(w_t)_{t\geq 0}$ be an exogenously given aggregate endowment process: each agent i has an endowment process $(\omega_t^i)_{t\geq 0}$ which is a stochastic share θ_t^i of the aggregate endowment, $\omega_t^i = \theta_t^i w_t$. The shares $(\theta_t^i)_{t\geq 0}$ are independent multiplicative random walks

$$\log \theta_{t+1}^i = \log \theta_t^i + \eta_{t+1}^i y_{t+1} - \frac{1}{2} y_{t+1}^2, \quad t \geq 0 \tag{18}$$

where η_{t+1}^i are standard normal random variables ($\eta_{t+1}^i \sim \mathcal{N}(0,1)$), independent across agents, independent across time, independent of the information $\mathcal{F}_t$ at date t, and independent of all other random variables in the economy such as the aggregate endowment or the payoffs of the securities: thus agents's income risks are not insurable. $(y_{t+1})_{t\geq 0}$ is an exogenously given stochastic process which determines the variability of the change in the agents' income shares from date t to $t+1$, since

$$\log \frac{\theta_{t+1}^i}{\theta_t^i} = \eta_{t+1}^i y_{t+1} - \frac{1}{2} y_{t+1}^2 \sim \mathcal{N}(-\frac{y_{t+1}^2}{2}, y_{t+1}^2) \tag{19}$$

It is easy to check that

$$E(e^{\eta_t^i y_t - \frac{1}{2}y_t^2}) = 1$$

so that

$$E(\theta_t^i) = E\left(e^{\sum_{s=0}^t (\eta_s^i y_s - \frac{1}{2}y_s^2)}\right) = 1$$

By an appropriate choice of the measure space of agents and an associated Law of Large Numbers (see footnote 5 in Chapter 14) it follows that $\sum_{i\in I}\theta_t^i = 1$ so that $\sum_{i\in I}\omega_t^i = w_t$. Thus by using the process (??) CD apply to an agent's share process θ_t^i the log normal process which is standard in the finance literature for describing an equity price process, with the added feature that the variability process y_t can be chosen in conjunction with the aggregate-output process: it can (for example) have a negative covariance with aggregate output, reflecting the idea that in bad times (low w_t) agents have greater uncertainty.

If each agent consumes his endowment, $x_t^i = \theta_t^i w_t$, then the personal valuation of agent i at date t for a security j with payoff V_{t+1}^j at date $t+1$ (dividend plus capital value if the security is long-lived) is

$$q_t^{i,j} = E\left[\delta\left(\frac{\theta_{t+1}^i w_{t+1}}{\theta_t^i w_t}\right)^{-\alpha} V_{t+1}^j \middle| \mathcal{F}_t\right]$$

which, in view of (19) can be written as

$$q_t^{i,j} = E\left[\delta\left(\frac{w_{t+1}}{w_t}\right)^{-\alpha} e^{-\alpha(\eta_{t+1}^i y_{t+1} - \frac{1}{2}y_{t+1}^2)} V_{t+1}^j \middle| \mathcal{F}_t\right]$$

Since η_{t+1}^i is independent of $\mathcal{F}_t$ and all other random variables, integrating with respect to η_{t+1}^i gives

$$q_t^{i,j} = E\left[\delta\left(\frac{w_{t+1}}{w_t}\right)^{-\alpha} e^{\frac{1}{2}\alpha(\alpha+1)y_{t+1}^2} V_{t+1}^j \middle| \mathcal{F}_t\right] \tag{20}$$

where $e^{\frac{1}{2}\alpha(\alpha+1)y_{t+1}^2}$ is the mean of $e^{-\alpha(\eta_{t+1}^i y_{t+1} - \frac{1}{2}y_{t+1}^2)}$ conditional on y_{t+1}. Since the right side of (20) is independent of i, all agents agree on the valuation of all securities (whose payoffs are independent of the idiosyncratic shocks η_t^i) and there are no gains to trade. Thus if security prices are given by (20), agents consume their initial endowments and there is no trade on financial markets ($z^i = 0,\ i \in I$), then we have an equilibrium of the above economy. The intuition underlying this no-trade equilibrium is that if an agent experiences a bad shock today, this permanently reduces his expected share of future aggregate output (all the agents have the same expected growth of consumption whatever the level of their consumption today) so that the agent is not in a position to borrow against future income. Similarly if some other agent experiences a good shock today, this permanently raises the expectation of the agent's future share of aggregate output so that the agent has no reason to save.

Introducing individual income processes which are non-stationary and heteroscedastic with conditional variance can be made to depend on aggregate output has the potential to give important new insights into the properties of equilibria with incomplet markets and the equity premium. Form the theoretical point of view, CD emphasize that any stochastic discount factor can be generated by an appropriate choice of the variability process: if M_t is a stochastic process and y_t is chosen so that

$$\left(\delta \frac{w_{t+1}}{w_t}\right)^{-\alpha} e^{\frac{1}{2}\alpha(\alpha+1)y_{t+1}^2} = \frac{M_{t+1}}{M_t}$$

the CD economy generates a discount factor (or pricing kernel in the language of finance) equal to M_t. However, the model is to replicate the stylized facts, the process y_t cannot be chosen arbitrarily. In an interesting recent paper, De Santis (2007) studied the conditions which need to be imposed on the processes w_t, y_t and D_t in the CD model, where D_t is the process of aggregate dividends traded in the stock market, in order that the equilibrium matches the level of interest rates, equity premium, volatility of stock prices, and the variability of the equity premium observed in US data. De Santis finds that permanence in agents' income shares does not suffice to match the data: a small persistent component in the dividend growth process needs to be introduced, which is sufficiently small to be compatible with the U.S data, but which generates volatility in stock prices. This volatility of the stock price generates sufficient risk in equity return to justify a sizable risk premium. The process y_t essentially serves to lower the interest rate via the precautionary motive of the agents with idiosyncratic risks, and a negative correlation of y_t with w_t explains the observed negative covariance of the equity premium with aggregate output.

A production version of the CD model has been developed by Krebs (2003) who, like Aiyagari, considers a standard growth model with idiosyncratic labor risks, with the difference that the labor endowment is viewed as a second form of capital—human capital—in which agents can invest and which is subject to idiosyncratic shocks in depreciation η_t^i, satisfying the same conditions as the shocks η_t^i of CD. Since the shocks affect the depreciation of human capital, they are permanent. Krebs finds that in equilibrium agents invest too much in physical capital (which has a sure return), too little in human capital (which is risky) and that, when the model is calibrated, it generates a significant excess return of the risky human capital over the sure return on physical capital, and a significant welfare loss due to incomplete markets.

Concluding Remarks

Most economists will probably agree that markets play an important role in co-ordinating economic activity over time. In the 1950's and 60's there was widespread agreement on what an equilibrium model of markets meant—it was the model with complete forward markets, known as the Arrow-Debreu model. In the 1980's an extension of this model to a setting where agents trade sequentially over time on a combined system of spot and financial markets came to be extensively studied, with particular emphasis on the idea that financial markets are incomplete: agents are exposed to many risks, not all of which can be optimally shared. Volume I has been devoted to the study of the development of this basic model over the last thirty years. In the 1990's the analysis of this model was extended to a dynamic setting over an infinite horizon: this Volume has examined this development of the model over the last ten years.

One of the interesting features of Arrow-Debreu theory and the GEI model is that they serve as a powerful link between a number of important areas of economic theory, in particular macroeconomics, microeconomics (markets) and finance. The key workhorse of macro—the Ramsey-Cass-Koopmans model of optimal growth—which evolved into the real business cycle model, is nothing but an infinite horizon representative agent Arrow-Debreu economy with production: in the same way a key workhorse of finance—the representative agent pricing model—is just an infinite horizon single agent Arrow-Debreu exchange economy. Both fields have explored the consequences of introducing imperfections (frictions) into these models and an especially important imperfection is that arising from the incompleteness of the markets.

The papers in this Volume have been chosen to give a feel for the development of dynamic GEI over the last ten years: the selection is inevitably very incomplete in view of the enormous parallel, and overlapping literature in macro and finance. The papers in Part I focused on giving a basic grammar of the infinite horizon GEI model by examining conditions for existence and the absence of Ponzi schemes, in models with or without bankruptcy. It was also shown that in a rational expectations equilibrium, bubbles on asset prices are unlikely to occur. Thus incompleteness of markets is not sufficient to explain the periodic speculative bubbles which occur in the real world on long-lived assets.

Just as in much of macro, steady states have provided an important starting point, not only for understanding the dynamic behavior of the models, but also for computing equilibria, so for the dynamic GEI model, stationary equilibria provided the simplest type of equilibrium not only for analyzing their properties, but also for computing the equilibria. The two papers in Part II provide a good starting point for understanding the basic problems involved with showing that stationary equilibria exist.

In a world where agents are exposed to stationary transient risks, holding a bufffer stock of a single asset or having access to borrowing and lending proves to be a powerful instrument for eliminating an agent's exposure to risk. We have referred ti the strategies that involve saving when the outcome is 'good' and borrowing or dissaving when the outcome is 'bad' as carry-over strategies. The papers in Part III show that carry-over strategies in a single asset are often sufficient to bring the equilibrium very close to the Arrow-Debreu equilibrium: this is either shown trough the technique of calibration typical of macroeconomics as in Chapters 9-11 and 13, or theoretically in the tradition of GEI as in Chapter 12. Thus even with incomplete markets, the equilibria have the properties pointed out by Mehra-Prescott, that interest rates are too high and the equity premia too low. In seeking to simplify the the dynamic GEI model and make the infinite horizon calculable, the risk process to which agents are subjected have been taken to be stationary, and what the papers in Chapters Hu-HL show, is that the assumption of stationarity in essence kills the effect of incomplete markets.

Already in Chapter 10, Aiyagari had pointed out that a high degree of permanence (autocorrelation) in an agent's labor income process makes it much harder to achieve consumption smoothing through carry-over strategies. Chapter 14 takes this observation to the limit and assumes that as agent income share process follow a multiplicative random walk: by making the shocks permanent agents can no longer smooth their consumption by carry-over strategies and we are back to the genuine 'incomplete markets' equilibrium. Thus in the dynamic GEI model, to ensure that the incompleteness of markets has a bit, the additional assumption that the agents' endowment processes are subject to sufficient permanence or non stationarity is necessary to end up with equilibria with imperfect risk sharing.

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