The following lemma shows that the foregoing counterexample is sharp.

Lemma 5.1 Let $N = \{1, 2\}$ and let A be a set of m members, $m \geqslant 2$. Furthermore, let Γ be an EGFPI and let R^N be a profile of weak orders. Then $g(\Gamma, R^N)$ has a PCPNE.

The proof is by induction on the number of moves in Γ. The details are omitted.

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14 Efficiency of Marginal Cost Pricing Equilibria

Martine Quinzii

1 INTRODUCTION

In an economy with increasing returns to scale the concept of a competitive equilibrium is no longer appropriate. What is the appropriate concept of equilibrium? No fully satisfactory answer has yet been given. Following the efficiency considerations put forward by Dupuit (1844) and Hotelling (1938), the early literature concentrated on a partial equilibrium analysis of marginal cost pricing by the sector with increasing returns. It was clear from the start that while such pricing might enhance productive efficiency, it created another problem, namely that of covering the costs of production. More subtle issues associated with marginal cost pricing only became apparent more recently when Guesnerie (1974) gave the first general equilibrium treatment of marginal cost pricing. In particular he provided a striking example, subsequently simplified by Brown and Heal (1982), of an economy with two goods, two consumers, one nonconvex firm and a rule for distributing income, which has two equilibria both of which are *inefficient*. To be sure, there exist some income distributions such that at least one associated equilibrium is efficient; this follows from the second welfare theorem which is still valid in non-convex economies (see Cornet, 1986; Guesnerie, 1975; Khan and Vohra, 1987; Quinzii, 1988; Yun, 1984). But this phenomenon greatly complicates the problem of finding an efficient allocation of resources. Not only must a planner control the nonconvex public sector and find a way of covering the losses created by marginal cost pricing, but he must also know *all* the characteristics of the economy in order to avoid those income distributions that lead to inefficient equilibria.

The possibility of obtaining inefficient marginal cost pricing equilibria is closely related to difficulties in establishing the *existence* of these equilibria. In particular the examples of Guesnerie (1975) and Brown and Heal (1982) mentioned above also show that the same imputation of utility can be obtained with two different production plans of the nonconvex sector even though the intermediate productions cannot achieve the same utility levels. This creates a discontinuity in the map from the utility frontier to the

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Pareto optimal allocations which prevents the use of Negishi's (1960) welfare approach for proving existence via a fixed point on the utility frontier.

In much of the recent literature, emphasis has been placed on finding conditions of progressively greater generality under which a marginal cost pricing equilibrium exists. The first existence results (Beato, 1976 and Mantel, 1979) were obtained for the case of economies with only one nonconvex firm, using a fixed point argument on the efficiency frontier of the production set. At first this argument was readily applicable only in the differentiable case.[1] Cornet (1982) subsequently showed that the appropriate way to extend the concept of marginal cost pricing to the case where the frontier exhibits kinks is via Clarke's concept of a normal cone. However, as Cornet (1982) also pointed out, the result obtained with an aggregate production set does not imply the existence of a decentralised equilibrium where individual firms apply the marginal cost pricing rule. An example provided by Beato and Mas-Colell (1983) made particularly clear the nature of the difficulty: they exhibited a simple economy with two firms producing the same good (one with increasing and one with constant returns to scale) in which no equilibrium is *productively efficient*. In addition to the further doubt that this example threw on the ability of the marginal cost pricing rule to lead to efficiency, it implied that an alternative strategy had to be found to prove existence of an equilibrium in the general case.

The problem of proving existence of an equilibrium with several firms motivated an active trend of research. Several authors extended the analysis to general pricing rules including different types of average cost pricing (see Beato and Mas-Colell, 1985; Bonnisseau and Cornet, 1988; Brown *et al.*, 1986; Dierker *et al.*, 1985; Kamiya, 1988; Vohra, 1988). Among the most general results of existence of marginal cost pricing equilibria obtained so far are those of Bonnisseau and Cornet (1990a-b). They show that only very mild assumptions are necessary to obtain the existence of marginal cost pricing equilibria.

Although the general equilibrium analysis has revealed that, with increasing returns, it is more difficult to achieve an efficient allocation of resources in a decentralised manner than the earlier partial equilibrium reasoning had suggested, a careful examination of the conditions under which the equilibria are efficient should be made before accepting or rejecting marginal cost pricing as an appropriate rule. If it is found that the circumstances under which marginal cost pricing leads to inefficiency are too pervasive, then a strong case can be made for applying a second best pricing scheme which has the advantage of being easier to implement. Despite its great practical importance, little attention has been devoted to finding conditions under which the first theorem of welfare holds for nonconvex economies. Two exceptions however should be noted. Brown and Heal (1981) have suggested two conditions under which one or all of the marginal cost pricing equilibria are Pareto-optimal. First, if the preferences are such that the consumer sector can be aggregated into a representative consumer then at least one of the marginal cost pricing equilibria is Pareto-optimal. The condition is clearly very restrictive. Second, if there exists an homeomorphism of the space of goods which transforms the economy (production sets and preferred sets) into a convex economy, then all equilibria are Pareto-optimal. No condition on technology and preferences has yet been found to ensure that this condition is satisfied. Dierker (1986) proposed conditions very similar in spirit to the ones presented in this study. I shall comment on them later.

In order to prevent the type of productive inefficiency exhibited by Beato and Mas-Colell, it may be necessary for the government to prevent the private sector for producing the goods that the public sector can produce with increasing returns. A careful coordination of the different units of the public sector can then achieve productive efficiency. In this paper we assume that this problem has been resolved and consider an aggregate public sector with a nonconvex technology set which is the sole producer of certain goods that we call the *public sector goods*.

An inefficient equilibrium occurs when the social indifference curve which is tangent to the feasible set at the equilibrium, reenters the feasible set because of the nonconvexity. To prevent this phenomenon it is natural to impose restrictions on the relative curvatures of the production frontier and of the social indifference curves. If regularity conditions are imposed on the public sector technology which restrict the nature of the nonconvexities, then we can use either the derivative or the elasticity of the marginal cost of production to measure the curvature of the production frontier. The derivative or the elasticity of the compensated demand measure the curvature of the social indifference curve. Our main theorem asserts that *if the product of the derivatives or of the elasticities evaluated at an equilibrium is less than 1 then the equilibrium is efficient*. As an immediate corollary we obtain that if the product is less than 1 in all circumstances, then *all marginal cost pricing equilibria are efficient*.

The model and the precise definitions of the functions that we consider – which involves mapping the original economy into a transformed two-good economy using the equilibrium prices – are presented in section 2. The theorems are presented in section 3 along with a geometric interpretation of the elasticity condition mentioned above; we also compare the results of this paper with the conditions given by Dierker for a similar model. Section 4 contains the proofs.

2 MODEL AND CONCEPT OF EQUILIBRIUM

Consider an economy in which the production of some of the goods involves substantial intervals of increasing returns to scale. Since selling these goods at marginal cost typically leads to a deficit, we will assume that the production of these goods is under the control of a planner who can impose marginal cost pricing and collect income from the competitive sector of the economy to cover the deficit. As mentioned above the planner's problem of coordinating the different units of the public sector to achieve an efficient production is not addressed here. We consider directly an aggregate public sector represented by a nonconvex production set. This sector coexists with a competitive sector composed of price taking consumers and firms. The competitive firms have convex production sets, maximise profit and do not produce the same goods as the public sector.

Public and Private Sector Goods

The $L + K$ commodities are indexed so that the first L goods are *private sector goods*; these are goods owned initially by the consumers or produced by the competitive firms. They are used for consumption or as inputs for the public sector but are not produced by the public sector. The remaining K goods are *public sector goods*. They do not exist initially in the economy and can be produced solely by the public sector. They can be used directly for consumption or indirectly as inputs for the competitive firms.

We use the following notation. An allocation for consumer i is denoted by (x^i, y^i) where $x^i \in R^L_+$, $y^i \in R^K_+$. A production plan of competitive firm j is denoted by (x_j, y_j), $x_j \in R^L$, $y_j \in R^K_-$ and a production plan of the public sector by (z, y), $z \in R^L_-$, $y \in R^K_+$.

The normalisation of the prices is as follows: a price vector is of the form $(p, \theta q)$ where p belongs to Δ_{L-1} (the simplex of R^L), q belongs to Δ_{K-1} and θ to R_+. p (resp. q) gives the relative prices of the private sector goods (resp. public sector goods) and θ indicates the level of the prices of the public sector goods relative to the prices of the private sector goods.

The Public Sector

The public sector is represented by a production set $T \subset R^L_- \times R^K$ with boundary ∂T which satisfies the following conditions.

P 1

(1) *T is closed.*
(2) $(0, 0) \in T$
(3) $(z, y) \in T$, and $(z', y') \leq (z, y) \Rightarrow (z', y') \in T$.

P 2 $\partial T \cap \{(z, y) \in R^{L+K} \mid y \gg 0\}$ is a smooth manifold of dimension $L + K - 1$.

P 2 implies that there is a unique tangent plane passing through each

$(z, y) \in \partial T$. We let $N_T(z, y)$ denote the positive half line normal to the tangent plane at (z, y)

P 3 $T_y = \{z \in R^L_- \mid (z, y) \in T\}$ is a nonempty, strictly convex set of R^L whose boundary is a smooth manifold with nonzero gaussian curvature, $\forall y \in R^K_{++}$.

P 3 implies that the cost function g: $\Delta_{L-1} \times R^K_+ \rightarrow R$ defined by

$$g(p, y) = inf\{-p \cdot z \mid z \in T_y\}$$

is a smooth function for $p \gg 0$ and $y \gg 0$. We assume moreover

P 4 The function: $y \rightarrow g(p, y)$ is increasing, strictly quasi-convex on R^K_{++}, with nonzero gaussian curvature, $\forall p \gg 0$.

Remark 1 We could weaken assumptions P 2 and P 3 to include the case where not all the competitive goods are factors of production of the public sector. This would simply require more notation. Assumptions P 2 and P 3 would then hold for the projection of T on $R^{L'} \times R^K$ where $L' \leq L$ is the number of inputs of the public sector. The reader can readily check that the analysis which follows is still valid in this case.

Remark 2 P 3 and P 4 severely restrict the nonconvexities of the production set T. P 3 is the standard assumption of convexity of the input requirement sets which is needed to obtain a well behaved cost function. P 4 requires the convexity of the production plans which cost less than a given amount: this assumption appears in partial equilibrium studies where multi-output firms are represented by their cost function. It follows from a simple application of the separation theorem for convex sets that P 3 and P 4 together imply the convexity of the sections $T_z = \{y \in R^K \mid (z, y) \in T\}$, the set of productions attainable with a given combination of inputs. The production set is thus assumed to be as well behaved as is compatible with the existence of increasing returns at some levels of production. No assumption of differentiability is made at $y = 0$ which allows for the existence of fixed costs.

P 3 and P 4 imply that the indirect cost function c: $\Delta_{L-1} \times \Delta_{K-1} \times R_+ \rightarrow R_+$ defined by

$$c(p, q, \eta) = \inf\{g(p, y) \mid q \cdot y \leq \eta\}$$

is an increasing, smooth function of η for $p \gg 0$, $q \gg 0$. Let $e(p, q, \eta)$

denote the *elasticity of the marginal cost* $\frac{\partial c}{\partial \eta}$ with respect to the level of production η

$$e(p, q, \eta) = \frac{\frac{\partial^2 c}{\partial \eta^2}(p, q, \eta)}{\frac{\partial c}{\partial \eta}(p, q, \eta)} \eta$$

We use the normalised price q to aggregate the outputs of the public sector into a level of production η. $c(p, q, \eta)$ then gives the minimum cost of the inputs required to produce outputs of a total value η. The function c is the cost function which would be used in a partial equilibrium model. We will see that under the convexity assumptions P 3 and P 4, the partial equilibrium approach can be extended to a general equilibrium analysis.

The Private Sector

(1) *Consumers* There are I consumers ($i = 1, \ldots, I$). The preferences of consumer i are represented by a utility function u^i: $R_+^{L+K} \to R_+$ and consumer i has an initial endowment w^i of the private sector goods, $w^i \in R_+^L$.

C 1 u^i is a smooth, monotonic, strictly quasi-concave function on R_{++}^{L+K} with nonzero gaussian curvature, $\forall i = 1, \cdots, I$. Whenever $x_h^i = 0$ or $y_h^i = 0$ for some good h then $u^i(x^i, y^i) = 0$.

C 1 implies that the compensated (Hicksian) demand of agent i

$$(\tilde{x}^i(p, \theta q, v^i), \tilde{y}^i(p, \theta q, v^i)) = \arg\min\{p \cdot x^i + \theta q \cdot y^i \mid (x^i, y^i) \in R_+^{L+K}, u^i(x^i, y^i) \geq v^i\}$$

is a smooth function for $(p, q, \theta, v^i) \in R_{++}^{L+K+2}$.

(2) *Competitive firms* There are J firms ($j = 1, \ldots, J$). Firm j has a convex production set T_j. Instead of making the assumptions on the production sets, for brevity we assume directly the properties of the supply function that we need:

C 2 Firm j has a supply function: $(p, \theta, q) \to (\tilde{x}^j(p, \theta q), \tilde{y}^j(p, \theta q))$ which is smooth on R_{++}^{L+K+1} and such that $\tilde{y}_j(p, \theta q) \leq 0, j = 1, \ldots, J$.

Assumptions C 1 and C 2 are made in order to obtain a differentiable demand function for the competitive sector. For a given imputation of utilities $v = (v^1, \ldots, v^I)$ and a price vector $(p, \theta q)$ the total compensated demand for the public sector goods is

$$\tilde{y}(p, \theta q, v) = \sum_{i=1}^{I} \tilde{y}^i(p, \theta q, v^i) - \sum_{j=1}^{J} \tilde{y}_j(p, \theta q)$$

We use the normalised price q to aggregate this demand to a level of demand that we denote by $\tilde{\eta}(p, \theta q, v)$. Thus

$$\tilde{\eta}(p, \theta q, v) = q \cdot \tilde{y}(p, \theta q, v)$$

The *elasticity of this aggregate demand* function with respect to the level θ of prices of the public sector is denoted by

$$\varepsilon(p, \theta q, v) = \frac{\frac{\partial \tilde{\eta}}{\partial \theta}(p, \theta q, v)}{\tilde{\eta}(p, \theta q, v)} \theta$$

Compensated Marginal Cost Pricing Equilibrium

To define a marginal cost pricing equilibrium we would need to close the model by a rule for distributing income in the economy. Private ownership of the private sector goods and firms is insufficient to provide such a rule since the deficit of the public sector has to be covered. For the purpose of this study, we do not need so much precision. Since we give conditions which imply that an equilibrium where all marginal rates of substitution are equalised is Pareto-optimal, the appropriate concept for us is the concept of a *compensated marginal cost pricing equilibrium* (CMCPE). Efficiency of marginal cost pricing equilibria can then be deduced from efficiency of CMCPE since, for any rule of income distribution, a marginal cost pricing equilibrium is always a CMCPE.

Definition A CMCPE is a vector $\{(\bar{x}^i, \bar{y}^i)_{i=1,\ldots,I}, (\bar{x}_j, \bar{y}_j)_{j=1,\ldots,J}, (\bar{z}, \bar{y}), (\bar{p}, \bar{q}, \bar{\theta}), \bar{v}\} \in R^{(L+K)(I+J+1)} \times R^{(L+K+1)} \times R^I$ such that

- $(\bar{x}^i, \bar{y}^i) = (\tilde{x}^i(\bar{p}, \bar{\theta}\bar{q}, \bar{v}^i), \tilde{y}^i(\bar{p}, \bar{\theta}\bar{q}, \bar{v}^i)), \quad i = 1, \ldots, I$ (14.1)
- $(\bar{x}_j, \bar{y}_j) = (\tilde{x}_j(\bar{p}, \bar{\theta}\bar{q}), \tilde{y}_j(\bar{p}, \bar{\theta}\bar{q})), \quad j = 1, \ldots, J$ (14.2)
- $(\bar{z}, \bar{y}) \in \partial T, (\bar{p}, \bar{\theta}\bar{q},) \in N_T(\bar{z}, \bar{y})$ (14.3)
- $\sum_{i=1}^{I} (\bar{x}^i - w^i) = \bar{z} + \sum_{j=1}^{J} \bar{x}_j$ (14.4)

- $\sum_{i=1}^{I} \bar{y}^i = \bar{y} + \sum_{j=1}^{J} \bar{y}_j$ (14.5)

Remark 3 Although the condition $(\bar{p}, \bar{\theta}\bar{q}) \in N_T(\bar{z}, \bar{y})$ does not directly involve the cost function, in the recent literature it is referred to as the *marginal cost pricing rule*. It is a geometric condition which ensures that the vector of prices is normal to the frontier of the production set. When the production set is convex this condition is, by the separation theorem, equivalent to profit maximisation. When the production set is nonconvex, we show that, under assumptions P 1–P 3, this condition implies that $\bar{z}$ minimises the cost of producing $\bar{y}$ at input prices $\bar{p}$ and that $\bar{\theta}\bar{q}$ is the vector of marginal costs. These assumptions, especially the assumption P 3 of convexity of the input requirement sets, are necessary to interpret condition (14.3) as a marginal cost pricing rule. Arrow and Hurwicz (1960) exhibited an example where P 3 does not hold and where condition (14.3) is satisfied at a point (z, y) but z does not minimise the cost of producing y. In this case, $\bar{\theta}\bar{q}$ cannot be interpreted as a vector of marginal costs. Let us show that under assumptions P 1–P 3, condition (14.3) corresponds to marginal cost pricing.

Lemma Under assumptions P 1–P 3, condition (14.3) implies:

- $\bar{z}$ minimises the cost of producing $\bar{y}$ at prices $\bar{p}$
- $\bar{\theta}\bar{q} = D_y g(\bar{p}, \bar{y})$

where $D_y g$ denotes the gradient of the cost function $y \rightarrow g(\bar{p}, y)$

Proof (see section 4)

Having clarified the meaning of a CMCPE, we are ready to state the main results of the paper.

3 CONDITIONS FOR EFFICIENCY OF EQUILIBRIA

Theorem 1 Let assumptions P 1–P 4, C 1, C 2 hold and let $\{(\bar{x}^i, \bar{y}^i), (\bar{x}_j, \bar{y}_j), (\bar{z}, \bar{y}), (\bar{p}, \bar{\theta}, \bar{q}), \bar{v}\}$ be a CMCPE. If either condition (i) or (ii) below holds,

$$\text{(i)} \quad \frac{\partial \tilde{\eta}}{\partial \theta}(\bar{p}, \theta\bar{q}, \bar{v}) \cdot \left. \frac{\partial^2 c}{\partial \eta^2}(\bar{p}, \bar{q}, \eta) \right|_{\eta=\tilde{\eta}(\bar{p}, \theta\bar{q}, \bar{v})} < 1 \quad \forall \theta > 0$$

$$\text{(ii)} \quad \varepsilon(\bar{p}, \theta\bar{q}, \bar{v}) \cdot e(\bar{p}, \bar{q}, \eta)|_{\eta=\tilde{\eta}(\bar{p}, \theta\bar{q}, \bar{v})} < 1 \quad \forall \theta > 0$$

then the CMCPE allocation is a Pareto-optimum.

Let V denote the set of feasible utility levels in the economy. Then as a corollary of Theorem 1 we have the following result:

Theorem 2 Let assumptions P 1–P 4, C 1, C 2 hold. If either condition (i) or (ii) above holds $\forall(\bar{p}, \bar{q}, \bar{v}) \in \Delta_{L-1} \times \Delta_{K-1} \times V$ then all CMCPE are Pareto-optimal.

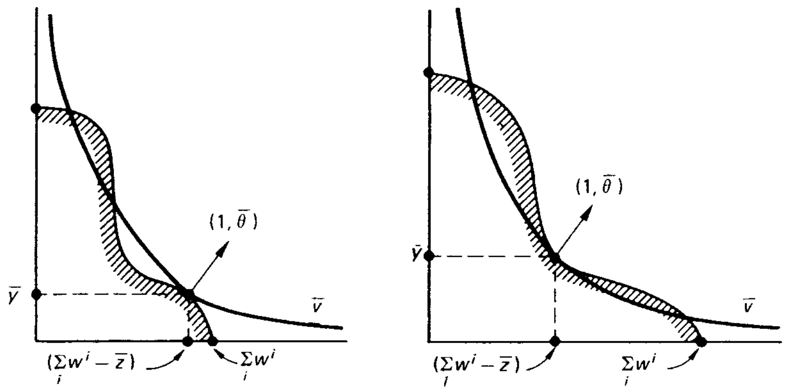
Remark 1 Theorem 1 gives conditions at an equilibrium which ensure that the equilibrium is optimal. Note that the conditions are readily checked by econometric methods. They require only a knowledge of the elasticity of the compensated demand when the public sector varies its price level while keeping the same relative prices for its outputs, and the elasticity of a crudely estimated cost function of the public sector with constant factor prices. Assuming that perfect competition prevails in the private sector and that the technology of the public sector satisfies P 3 and P 4 a planner could easily verify from the elasticity condition if a prevailing equilibrium was Pareto-optimal.

Remark 2 Theorem 2 gives global conditions which ensure that all marginal cost pricing equilibria are Pareto-optimal *whatever the rule chosen for financing the public sector*. If an economy satisfies these conditions then the phenomenon exhibited by Guesnerie, that some rules of income distribution lead to inefficient equilibria, cannot arise. A planner who can use direct, nondistortive taxation to finance the public sector may have to consider issues of equity in the implied redistribution of income but does not have to worry about efficiency. Marginal cost pricing and direct taxation always lead to an efficient outcome.

Although the assumptions of Theorem 2 are restrictive they are satisfied for most of the simple models that could be built to study the property of marginal cost pricing. In Quinzii (1982; 1988) it is shown that for an economy composed of an aggregate nonconvex public sector and consumers, if the consumers have (possibly different) Cobb–Douglas utility functions and if the nonconvex sector produces one output with either a Cobb–Douglas or a CES production function with increasing returns, the assumptions of Theorem 2 are satisfied.

The proof of Theorem 1 is given in section 4. Here we explain the geometric intuition which underlies the conditions (i) and (ii) in the case of a simple two-good model.

Consider an economy composed of consumers and of a nonconvex, one input–one output firm. Let $\{(\bar{x}^i, \bar{y}^i), (\bar{z}, \bar{y}), (1, \bar{\theta}), \bar{v}\}$ be a CMCPE. The associated allocation is not Pareto-optimal if the social indifference curve corresponding to the imputation $\bar{v}$ reenters the feasible set – the translation by the total initial endowment of the production set – as shown in Figure 14.1a or 14.1b.



(a) (b)

Figure 14.1

With only two goods, (i) can be written

$$c''(1, \tilde{y}(1, \theta, \bar{v})) \cdot \frac{\partial \tilde{y}}{\partial \theta}(1, \theta, \bar{v}) < 1 \quad \forall \theta > 0$$

This is equivalent to

$$\frac{\partial c'}{\partial \theta}(1, \tilde{y}(1, \theta, \bar{v})) < 1$$

which implies that the function $\theta \to c'(1, \tilde{y}(1, \theta, \bar{v})) - \theta$ is decreasing.

Condition (ii) is

$$\frac{c''(1, \tilde{y}(1, \theta, \bar{v}))}{c'(1, \tilde{y}(1, \theta, \bar{v}))}\tilde{y}(1, \theta, \bar{v})\frac{\frac{\partial \tilde{y}}{\partial \theta}(1, \theta, \bar{v})}{\tilde{y}(1, \theta, \bar{v})}\theta < 1$$

which is equivalent to

$$c''(1, \tilde{y}(1, \theta, \bar{v}))\frac{\partial \tilde{y}}{\partial \theta}\theta - c'(1, \tilde{y}(1, \theta, \bar{v})) < 0$$

This implies that the function $\theta \to \dfrac{c'(1, \tilde{y}(1, \theta, \bar{v}))}{\theta}$ is decreasing.

As $c'(1, \tilde{y}(1, \bar\theta, \bar{v})) = \bar\theta$, both conditions imply that

$$\begin{cases} c'(1, \tilde{y}(1, \theta, \bar{v})) < \theta, & \text{if } \theta > \bar\theta \\ c'(1, \tilde{y}(1, \theta, \bar{v})) > \theta, & \text{if } \theta < \bar\theta \end{cases}$$

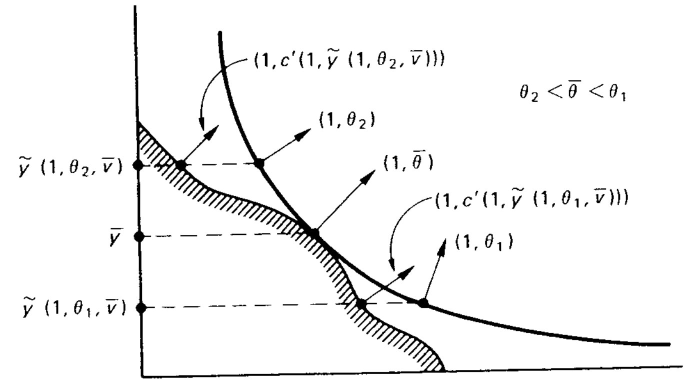


Figure 14.2

Thus, as shown in Figure 14.2, the social indifference curve is more curved at the equilibrium than the frontier of the production set and lies entirely above the feasible set.

It is interesting to note the relation between the Theorems 1 and 2 and the related results of Dierker (1986). The model studied by Dierker is the same as the one considered here with a competitive sector satisfying C 1 and C 2 and a public sector satisfying P 1–P 4. P 4 is expressed by Dierker as

P 4' $\{(z, y) \in T \mid q \cdot y = \eta\}$ is convex $\forall q \in \Delta_{K-1}$ $\forall \eta > 0$

One can prove that P 4' is equivalent to the quasi-convexity of $y \to g(p, y)$. The main difference is that Dierker makes separate assumptions on the private and the public sector. For the competitive sector his condition is equivalent to the elasticity of the function $\theta \to \tilde{\eta}(\bar{p}, \theta\bar{q}, \bar{v})$ being greater than -1. For the public sector, as he does not introduce the indirect cost function $c(p, q, \eta)$, the condition is more involved. One has to follow the curve on the production frontier passing through $(\bar{z}, \bar{y})$ consisting of the points where the normal vector is of the form $(\bar{p}, \theta\bar{q})$. Parametrising the curve by the value $\eta = \bar{q} \cdot y$, $\theta(\eta)\bar{q} \cdot y(\eta)$ has to be an increasing function of η. One can prove that this condition is equivalent to the elasticity of $c'(\bar{p}, \bar{q}, \eta)$ being greater than -1. Thus the conditions of Dierker are more restrictive than (ii). In particular one would expect that for large values of θ, the absolute value of the elasticity of $\tilde{\eta}$ becomes greater than 1. This would not violate condition (ii) provided that the elasticity of the marginal cost stays small for small values of production.

Dierker arrived at his conditions by mapping the original economy into an

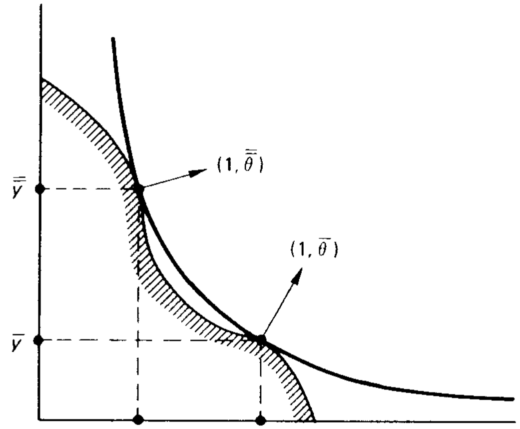


Figure 14.3

artificial economy with a public good, whereas the conditions (i) and (ii) were first introduced to prove existence of the core for an economy with increasing returns (Quinzii, 1982; Ichiishi and Quinzii, 1983). Since (i) and (ii) imply that the social indifference curve tangent to the feasible set at an equilibrium is above the feasible set, it prevents the case shown in Figure 14.3.

In this case there are two levels of production $\bar{y}$, $\bar{\bar{y}}$ which lead to the same imputation $\bar{v}$; however the intermediate levels of production fail to achieve $\bar{v}$. A small perturbation of v in the neighbourhood of $\bar{v}$ would result in a discontinuity at $\bar{v}$ of the map $v \rightarrow \{(x^i, y^i), (z, y)\}$ from the Pareto frontier in the utility space to the Pareto frontier in the allocation space.

When the assumptions of Theorem 2 hold, this discontinuity cannot occur and the Negishi approach (a fixed point argument on the frontier of the feasible imputations) can be applied to prove existence of a CMCPE. The fixed point argument can be modified, in the case where the production set T exhibits increasing returns everywhere, to prove existence of the core (see Quinzii, 1988 for the complete proofs). The assumptions of Theorem 2 in conjunction with increasing returns in the public sector thus define a class of economies which have almost all the properties of convex economies. For any rule of distribution of income there exists an equilibrium. Equilibria are efficient and the core of the economy is not empty. The only missing element is a canonical rule for distributing income (supplementing the private ownership of resources) which would lead to equilibria in the core.

4 PROOFS

Proof of Lemma Let $((\bar{z}, \bar{y})\ (\bar{p}, \bar{\theta}, \bar{q}))$ be such that

$$(\bar{z}, \bar{y}) \in \partial T, \quad \bar{y} \gg 0, \quad (\bar{p}, \bar{\theta}\bar{q}) \in N_T(\bar{z}, \bar{y}) \tag{14.4.1}$$

Since every manifold of R^{H+K} is locally expressible as a graph, there exists, by P 2, a neighbourhood V of $(\bar{z}, \bar{y})$ and a function f: $R^{H+K} \rightarrow R$ such that if $(z, y) \in V$, $(z, y) \in \partial T$ is equivalent to $f(z, y) = 0$. The condition $(\bar{p}, \bar{\theta}\bar{q}) \in N_T(\bar{z}, \bar{y})$ is then equivalent to

$$(\bar{p}, \bar{\theta}\bar{q}) = \bar{\lambda} Df(\bar{z}, \bar{y}) \tag{14.4.2}$$

for some positive real number $\bar{\lambda}$. $\bar{z}$ belongs to $\partial T_{\bar{y}}$ which can be represented locally around $\bar{z}$ by the equation $f(z, \bar{y}) = 0$.

Consider the programme

$$\min\{-\bar{p} \cdot z \mid z \mid f(z, \bar{y}) = 0\} \tag{14.4.3}$$

Equation (14.4.2) implies $\bar{p} = \bar{\lambda} D_z f(\bar{z}, \bar{y})$ so that the first-order conditions of the programme (14.4.3) are satisfied at $\bar{z}$. The first-order conditions of the programme $\min\{-\bar{p} \cdot z \mid z \in T_{\bar{y}}\}$ are therefore satisfied at $\bar{z}$ and, since this programme is, by P 3, convex

$$g(\bar{p}, \bar{y}) = -\bar{p} \cdot \bar{z}$$

Since by P 3 the function $z \rightarrow f(z, y)$ has nonzero gaussian curvature at $\bar{z}$ we can apply the implicit function theorem to the system of equations

$$\begin{cases} \bar{p} = \lambda D_z f(z, y) \\ f(z, y) = 0 \end{cases}$$

For y in a neighbourhood of $\bar{y}$, let $z(y)$ denote the solution of these equations. The function $y \rightarrow z(y)$ is smooth and the same argument as above proves that $g(\bar{p}, y) = -\bar{p} \cdot z(y)$. Then

$$\frac{\partial g}{\partial y_k}(\bar{p}, y) = -\bar{p} \cdot \frac{\partial z}{\partial y_k} = -\lambda D f_z(z(y), y) \cdot \frac{\partial z}{\partial y_k} = \lambda \frac{\partial f}{\partial y_k}(z(y), y)$$

Thus by equation (14.4.2)

$$D_y g(\bar{p}, \bar{y}) = \bar{\theta}\bar{q}$$

Proof of Theorem 1 Let $\{(\bar{x}^i, \bar{y}^i), (\bar{x}_j, \bar{y}_j), (\bar{z}, \bar{y}), (\bar{p}, \bar{\theta}, \bar{q})\}$ be a CMCPE. The assumptions on preferences imply that $(\bar{p}, \bar{\theta}, \bar{q}) \gg 0$ and that $\bar{v}^i = 0$ is possible only with $w^i = 0$ in which case agent i does not participate in the economy. We can therefore assume that $\bar{v}^i > 0 \quad \forall i = 1, \ldots, I$.

Consider the function V: $(0,+\infty) \to R_+$ defined by:

$$V(\eta) = \inf \left\{ \bar{p} \cdot \left(\sum_{i=1}^{I} x^i - \sum_{j=1}^{J} x_j \right) + c(\bar{p}, \bar{q}, \eta) \;\middle|\; \begin{array}{l} u^i(x^i, y^i) \geqslant \bar{v}^i \quad i = 1, \ldots, I \\ (x_j, y_j) \in T \quad j = 1, \ldots, J \qquad \bar{q} \cdot \left(\sum_i y^i - \sum_j y_j \right) = \eta \end{array} \right\}$$

When $\eta > 0$ is fixed, the programme which defines $V(\eta)$ is convex and has a unique solution given by the first-order conditions. Let $(\lambda^i)_{i=1,\ldots,I}$ be the multiplier associated with the first I constraints, θ the multiplier associated with the last one. The first-order conditions are:

$$\frac{\partial u^i}{\partial x_l^i} = \lambda^i \bar{p}_l \quad i = 1, \ldots, I \quad l = 1, \ldots, L \tag{14.4.4}$$

$$\frac{\partial u^i}{\partial y_k^i} = \lambda^i \theta \bar{q}_k \quad i = 1, \ldots, I \quad k = L+1, \ldots, L+K \tag{14.4.5}$$

$$u^i(x^i, y^i) = \bar{v}^i \quad i = 1, \ldots, I \tag{14.4.6}$$

$$(\bar{p}, \theta\bar{q}) \in N_{T_j}(x_j, y_j) \quad j = 1, \ldots, J \tag{14.4.7}$$

$$\bar{q} \cdot \left(\sum_{i=1}^{I} y^i - \sum_{j=1}^{J} y_j \right) = \eta \tag{14.4.8}$$

Let us solve the system of equation as follows: the allocation which solves equations (14.4.4)–(14.4.6) is $(\tilde{x}^i(\bar{p}, \theta\bar{q}, \bar{v}^i), \tilde{y}^i(\bar{p}, \theta\bar{q}, \bar{v}^i))_{i=1,\ldots,I}$. Since the production sets T_j are convex, condition (14.4.7) is equivalent to profit maximisation and the allocation which solves equation (14.4.7) is $(\tilde{x}_j(\bar{p}, \theta\bar{q}), \tilde{y}_j(\bar{p}, \theta\bar{q}))_{j=1,\ldots,J}$. Substituting these expressions into (14.4.8) gives the following equation for determining θ

$$\bar{q} \cdot \left(\sum_i \tilde{y}^i(\bar{p}, \theta\bar{q}, \bar{v}^i) - \sum_j \tilde{y}_j(\bar{p}, \theta\bar{q}) \right) = \eta \tag{14.4.9}$$

which can be rewritten as

$$\tilde{\eta}(\bar{p}, \theta\bar{q}, \bar{v}) = \eta \tag{14.4.9'}$$

Under the assumptions C 1 and C 2 the function $\tilde{\eta}$ is decreasing. In fact, denoting $\bar{\pi} = \theta\bar{q}$ the vector of prices of the K outputs of the public sector

$$\frac{\partial\tilde{\eta}}{\partial\theta}(\bar{p}, \theta\bar{q}, \bar{v}) = \sum_{k=L+1}^{L+K} \sum_{k'=L+1}^{L+K} \bar{q}_k \bar{q}_{k'} \left(\sum_i \frac{\partial\tilde{y}_k^i}{\partial\pi_{k'}} - \sum_j \frac{\partial\tilde{y}_{jk}}{\partial\pi_{k'}} \right)$$

$$= \frac{1}{\theta^2} \left\langle (0, \bar{\pi}), \frac{\partial(\tilde{x}, \tilde{y})}{\partial(p, \pi)} \cdot (0, \bar{\pi}) \right\rangle$$

where $\frac{\partial(\tilde{x}, \tilde{y})}{\partial(p, \pi)}$ denotes the matrix of partial derivatives of the aggregate compensated demand of the competitive sector with respect to the prices. This matrix is negative semidefinite as a sum of negative semidefinite matrices and the assumptions on the preferences imply that its kernel contains only the vectors colinear to the price vector $(\bar{p}, \bar{\pi})$ (see for example Guesnerie (1975 p. 21). As $(0, \bar{\pi})$ is not colinear to $(\bar{p}, \bar{\pi})$, $\frac{\partial\tilde{\eta}}{\partial\theta}(\bar{p}, \theta\bar{q}, \bar{v}) < 0$.

Thus, for η in the range of the function $\tilde{\eta}$, equation (14.4.9′) has a unique solution $\tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \eta)$ and the envelope theorem implies

$$\frac{\partial V}{\partial\eta}(\eta) = -\tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \eta) + \frac{\partial c}{\partial\eta}(\bar{p}, \bar{q}, \eta) \tag{14.4.10}$$

Since we have proved that $\frac{\partial\tilde{\eta}}{\partial\theta} < 0$, the implicit function theorem implies that $\tilde{\theta}$ is differentiable in η and

$$\frac{\partial^2 V}{\partial\eta^2}(\eta) = -\frac{\partial\tilde{\theta}}{\partial\eta}(\bar{p}, \bar{q}, \bar{v}, \eta) + \frac{\partial^2 c}{\partial\eta^2}(\bar{p}, \bar{q}, \eta)$$

By equation (14.4.9′)

$$\frac{\partial^2 V}{\partial\eta^2}(\eta) = \frac{-1 + \frac{\partial^2 c}{\partial\eta^2}(\bar{p}, \bar{q}, \eta)\frac{\partial\tilde{\eta}}{\partial\theta}(\bar{p}, \tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \eta)\bar{q}, \bar{v})}{\frac{\partial\tilde{\eta}}{\partial\theta}(\bar{p}, \tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \eta)\bar{q}, \bar{v})} \tag{14.4.11}$$

Condition (i) implies that $\frac{\partial^2 V}{\partial\eta^2}(\eta) > 0$ for all η in the range of the function $\tilde{\eta}$ and thus V is convex. Condition (ii) implies that $\frac{\partial^2 V}{\partial\eta^2}(\eta) > 0$ whenever $\frac{\partial V}{\partial\eta}(\eta) = 0$. In both cases, the condition $\frac{\partial V}{\partial\eta}(\eta) = 0$, i.e.

$$\tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \eta) = \frac{\partial c}{\partial \eta}(\bar{p}, \bar{q}, \eta) \tag{14.4.12}$$

is necessary and sufficient for a global minimum of V.

Let $\bar{\eta} = \bar{q}\cdot\bar{y}$ be the value of the production at the CMCPE. The equilibrium condition on the markets of the public sector goods implies that $\tilde{\eta}(\bar{p}, \bar{\theta}\bar{q}, \bar{v}) = \bar{\eta}$ so that, by equation (14.4.9′),

$$\bar{\theta} = \tilde{\theta}(\bar{p}, \bar{q}, \bar{v}, \bar{\eta}) \tag{14.4.13}$$

In the preceding lemma it has been proved that $\bar{\theta}\bar{q} = Dg_y(\bar{p}, \bar{y})$. This implies that the first-order conditions of the programme

$$\inf\{g(\bar{p}, y) \mid \bar{q} \cdot y = \bar{\eta}\}$$

are met at $\bar{y}$. By assumption P 4, $\bar{y}$ is the unique solution of this programme and $\bar{\theta}$ is the multiplier associated with the constraint. Thus

$$\begin{aligned} c(\bar{p}, \bar{q}, \bar{\eta}) &= g(\bar{p}, \bar{y}) \\ &= \bar{p} \cdot \bar{z} \text{ (by the Lemma)} \end{aligned}$$

and by the envelope theorem

$$\frac{\partial c}{\partial \eta}(\bar{p}, \bar{q}, \bar{\eta}) = \bar{\theta} \tag{14.4.14}$$

Equations (14.4.13) and (14.4.14) imply that (14.4.12) holds at $\bar{\eta}$. Thus $\min_\eta V(\eta) = V(\bar{\eta})$. $((\bar{x}^i, \bar{y}^i), (\bar{x}_j, \bar{y}_j), \bar{\theta})$ verify equations (14.4.4)–(14.4.8) for $\bar{\eta}$ so that

$$V(\bar{\eta}) = \bar{p} \cdot \left(\sum_i \bar{x}^i - \sum_i \bar{x}_j\right) + c(\bar{p}, \bar{q}, \bar{\eta}) = \bar{p} \cdot \sum_i w^i$$

Therefore

$$\min_\eta V(\eta) = \bar{p} \cdot \sum_i w^i \tag{14.4.15}$$

Suppose that the CMCPE is not Pareto-optimal. Then there exists a feasible allocation $((\hat{x}^i, \hat{y}^i), (\hat{x}_j, \hat{y}_j), (\hat{z}, \hat{y}))$ such that $u^i(\hat{x}^i, \hat{y}^i) > \bar{v}^i$ $\forall i = 1, \ldots, I$. Let $\hat{\eta} = \bar{q} \cdot \hat{y}$ be the value of the production at prices $\bar{q}$. By assumption C 1 on preferences, there would also exist $\hat{\hat{x}}^i \ll \hat{x}^i$ such that $u^i(\hat{\hat{x}}^i, \hat{y}^i) > \bar{v}^i$. Thus

$$V(\hat{\eta}) < \bar{p} \cdot \left(\sum_i \hat{x}^i - \sum_j \hat{x}_j\right) - \bar{p} \cdot \hat{z} \leqslant \bar{p} \cdot \sum_i w^i$$

which contradicts equation (14.4.15).

Note

1. Although Beato (1976; 1982) extended the result to the case of a nondifferentiable production frontier using the cone of interior displacements. But the cone of interior displacements is not always convex and restrictive assumption on the sort of 'kinks' which are allowed in the production frontier are necessary to obtain the existence result.

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15 A Long-open Question on Utility and Conserved-energy Functions*

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1 INTRODUCTION

Suppose I am indifferent between 'even–odds probability of getting x or y dollars' and 'the certainty of getting $z = f(x, y)$ dollars'.

Then $f(x, y)$ is a *symmetric mean*

$$f(x, x) \equiv x, \quad f(x, y) \equiv f(y, x) \tag{15.1}$$

It can be specified to be a single-valued, *continuous* function of its arguments. Also it can be supposed to be *strictly increasing* in either variable:

$$f(a, y) > f(b, y) \quad \text{iff } a > b \tag{15.2}$$

Sometimes one assumes additionally that $f(x, y)$ is smoothly twice differentiable

$$\begin{aligned} &\partial f(x, y)/\partial x \text{ is a well-defined continuous function} \\ &\partial^2 f(x, y)/\partial x^2 \text{ and } \partial^2 f(x, y)/\partial x \partial y \text{ are continuous functions} \end{aligned} \tag{15.3}$$

By definition, $f(x, y)$, a symmetric mean, is an *associative* (or 'quasi-linear') *mean* iff there exists a $F[z]$ function such that

$$F[f(x, y)] \equiv \tfrac{1}{2}F[x]+\tfrac{1}{2}F[y] \tag{15.4a}$$

$F[z]$ is a single-valued continuous function

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