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An Existence Theorem for the Core of a Productive Economy with Increasing Returns

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INTRODUCTION

In this paper, we derive a set of sufficient conditions for the existence of the core of a productive economy with increasing returns.

We consider a model in which the productive sector is described by a set of non-producible inputs, a set of produced outputs and a technology with increasing returns to scale. The economic agents have initial endowments of non-producible goods and preferences on the space of commodity vectors. No institutional organization of the production is specified but it is assumed that the technology is available to all. This means that any group of agents has access to the same technology and can undertake production for its own benefit, taking the inputs from its initial resources and consuming the outputs.

Within this framework, the core of the economy consists of feasible allocations for the grand coalition, which are therefore the result of an organization of production in a single unit. In addition, they have the property that no subset of agents can improve the welfare of its members by producing and consuming in autarcy. In presence of economies of scale, intuition suggests that the best opportunities for all the agents will occur if production is carried out in a single unit of the largest possible size. The possible existence of core allocations provides a test for this intuition which underlies all the literature on increasing returns. If the core is empty, then there will always exist a subset of agents who will find a common interest to break away from the great coalition and organize their own production in a smaller unit.

The core of such an economy has been studied mainly by Scarf [5–7]. He proved that, within a two goods economy (one input, one output), the assumption of increasing returns in production implies the nonemptiness of the core. Sharkey [9] pointed out that the same proof can be applied to the one input–several outputs model under the assumption that the cost function has "Decreasing Average Cost" (see Sharkey [9] for the definition).

But Scarf [7] proved also that, as soon as there are more than one input, one can associate with every production set with increasing returns, an economy whose core is empty. This result implies that, when there are several inputs, conditions for the non-emptiness of the core must bear simultaneously on the production side and the consumption side of the economy. Scarf [6] has given an example of such a condition: if some inputs do not enter the consumers preferences and if the production set is distributive with respect to these goods (see Scarf [6] for the definition), then, for any choice of utility functions and initial endowments, the economy has a non-empty core.

In this paper, we assume that there exists several inputs and that all commodities enter the preferences of the agents. But we restrict the study to the one output case. We obtain the result that, if there is no level of production for which the marginal cost is "too abruptly" decreasing, then the core is non-empty. (The meaning of "too abruptly" depends on the consumption side of the economy and will become precise in Section III.) This result may be interpreted as a continuity result: if there are constant returns to scale (i.e., constant marginal cost), it is well known that the core of the economy is non-empty. If returns to scale are increasing, but the variation of marginal cost is not too large, this property still holds.

The paper is organized as follows. The model and the assumptions are presented in Section I. In Section II, we introduce a sufficient condition, denoted condition (NE), which implies the non-emptiness of the core. Its introduction is rather natural from a technical point ofview, but its economic meaning is not clear. For this reason, we derive, in Section III, a set of conditions which implies the condition (NE) and to which one can easily give an economic interpretation.

I. The Model

Let us consider an economy with:

— $l+1$ goods. The first l goods (indexed by $h \in [1 \cdots l]$ are non-producible goods which can either be consumed or used in production as inputs. The last good is the only output of production and does not exist initially in the economy.

— n agents (indexed by $i \in [1 \cdots n]$). The set of all the agents is denoted N.

Agent i has a consumption set $X_i = \mathbb{R}_+^{l+1}$ on which he has a preference

preordering which can be represented by a utility function u_i. We will consider the following assumptions on the utility functions u_i:

(A1) Continuity: u_i is a continuous function from $\mathbb{R}_+^{l+1}$ onto $\mathbb{R}_+$.

(A2) Strict monotonicity: $\forall \xi_i \in \mathbb{R}_{++}^{l+1}$, $\forall \xi_i^1 \in \mathbb{R}_{++}^{l}$,

$$\xi_i \geqslant \xi_i', \qquad \xi_i \neq \xi_i' \Rightarrow u_i(\xi_i) > u_i(\xi_i').$$

(A3) Boundary condition: $\forall v_i > 0$, the closure of the indifference hypersurface $u_i^{-1}(v_i)$ in $\mathbb{R}^{l+1}$ is contained in $\mathbb{R}_{++}^{l+1}$.

(A4) Strict quasi-concavity: u_i strictly quasi-concave on $\mathbb{R}_{++}^{l+1}$.

(A5) u_i is C^2 on $\mathbb{R}_{++}^{l+1}$.

Agent i has initial resources $\omega_i \in \mathbb{R}_+$. We assume that

$$\omega = \sum_{i=1}^{n} \omega_i \gg 0.$$

— A production set $Y \subset \mathbb{R}^l \times \mathbb{R}_+$.

An element of Y will be denoted (z, y), where $z \in \mathbb{R}_-^l$ is the vector of inputs, $y \in \mathbb{R}_+$ is the output.

We consider the following assumptions on Y.

(P1) · Y is non-empty and closed,
· $(0, 0) \in Y$,
· $\forall (z, y) \in Y$, $z' \leqslant z$, $y' \leqslant y \Rightarrow (z', y') \in Y$,
· $\forall z \in \mathbb{R}_-^n$, $\{y \mid (z, y) \in Y\}$ is bounded.

(P2) Y presents non-decreasing returns to scale:

$$\forall (z, y) \in Y, \qquad \forall \lambda \geqslant 1 \qquad (\lambda z, \lambda y) \in Y.$$

(P3) The sections of Y are strictly convex and decreasing:

$$\forall y > 0, \qquad S_Y(y) = \{z \in \mathbb{R}_-^l \mid (z, y) \in Y\}$$

is non-empty and strictly convex:

$$\forall y > y', \qquad S_Y(y) \subset \dot{S}_Y(y')^1.$$

If we give a price system for the inputs, we can define a cost function for the output in the following way:

Let $\Delta = \{p \in \mathbb{R}_+^l, \sum_{h=1}^{l} p_h = 1\}$ be the simplex of input prices

$$\dot{\Delta} = \{p \in \Delta \mid p \gg 0\}.$$

[1] $\dot{S}_Y(y)$ is the interior of $S_Y(y)$ in $\mathbb{R}^l$.

For each price p, the cost of a quantity y of output is:

$$g(p, y) = \inf\{-p \cdot z \mid (z, y) \in Y\}.$$

The asumptions P1, P2, P3 on Y imply that:

(C1) $\forall p \in \Delta$, $\forall y \geqslant 0$, $g(p, y) \geqslant 0$; $y = 0 \Rightarrow g(p, y) = 0$,

(C2) $\forall p \in \mathring{\Delta}$, $g(p, \cdot)$ is strictly increasing and $\lim_{y \to +\infty} g(p, y) = +\infty$.

(C3) $\forall p \in \Delta$, $g(p, y)$ is subhomogeneous:

$$g(p, \lambda y) \leqslant \lambda g(p, y), \qquad \forall \lambda \geqslant 1.$$

The strict convexity of $S_Y(y)$ implies that, if $p \gg 0$, there is a unique input combination which minimizes the cost of a quantity y of output. We will denote $-\zeta(p, y)$ this vector.

$$\forall p \in \mathring{\Delta}, \qquad -p \cdot \zeta(p, y) = g(p, y).$$

We will need assumptions on the regularity of the set Y. These assumptions will be expressed directly on g and ζ rather than on Y.

(C4) g is continuous on $\Delta \times \mathbb{R}_+$.

(C5) ζ is continuous on $\mathring{\Delta} \times \mathbb{R}_+$.

(C6) $\forall p \in \mathring{\Delta}$, $y \to g(p, y)$ is C^2 on $\mathbb{R}_+$.

An allocation for this economy is a vector $\xi = (x_i, y_i)_{1 \leqslant i \leqslant n}$ belonging to $\mathbb{R}_+^{(l+1)n}$. $\xi_i = (x_i, y_i)$ represents the consumption of agent i.

An allocation ξ is feasible if $(\sum_{i=1}^n (x_i - \omega_i), \sum_{i=1}^n y_i) \in Y$.

A feasible allocation ξ is in the core of the economy if no coalition can block it. As told in the introduction we assume that each coalition has access to the production set Y. Thus, allocation ξ is blocked by a coalition S if there exists an allocation ξ' such that

$$\cdot\ u_i(\xi_i') > u_i(\xi_i), \qquad \forall i \in S,$$

$$\cdot\ \left(\sum_{i \in S} (x_i' - \omega_i), \sum_{i \in S} y_i'\right) \in Y.$$

A feasible allocation is Pareto optimal if it is not blocked by the coalition N of all the agents. The set of Pareto optimal allocations will be denoted P.

An imputation is a vector $v = (v_1 \cdots v_n)$ of $\mathbb{R}_+^n$. v_i represents the utility level of agent i.

The set of Pareto optimal imputations (the image of P by the function $u = (u_1 \cdots u_n)$) will be denoted $P(u)$.

II. The Game $\varphi(p, v, \cdot)$

To study the core of an economy, it is usual to consider the non-transferable utility game, which associates to each coalition the utility levels that it can ensure to its members by using the sum of their initial resources. The core of the economy is non-empty if and only if the core of this game is non-empty. The results of Scarf [5], Sharkey [9] are obtained by studying this canonical game.

Here, we will use a game which can be considered as a "dual" game, which associates, to each coalition S and each imputation v, the minimum cost for S of the imputation v. This game was first introduced by Champsaur [2] to find allocations in the core of a public good economy. We will prove in Section II that if condition (NE) (see below) holds, we can find by this method an allocation in the core of the economy described in Section I, which is therefore non-empty. In Section III we will give a set of economically meaningful conditions which imply this technical condition (NE) and then the existence of the core (Theorem 10).

Let $p \in \Delta$ be a price for the inputs and $v \in \mathbb{R}_+^n$ an imputation. We define:

$$\cdot\ U(v, S) = \{\xi \in \mathbb{R}_+^{(l+1)n} \mid u_i(\xi_i) \geqslant v_i\ \forall i \in S\}.$$

$U(v, S)$ is the set of the allocations giving at least the utility level v_i to the agents of the coalition S.

$$\cdot\ c(p, \xi, S) = p \cdot \sum_{i \in S} x_i + g\left(p, \sum_{i \in S} y_i\right).$$

$c(p, \xi, S)$ is the cost of an allocation ξ for a coalition S.

$$\cdot\ \varphi(p, v, S) = \inf\{c(p, \xi, S) \mid \xi \in U(v, S)\}.$$

$\varphi(p, v, S)$ is the minimum cost for a coalition S to give to its members the utility level v.

We will be interested in the allocation realizing the minimum of cost—when such an allocation exists. So we define

$$\xi(p, v, S) = \{\xi \in U(v, S) \mid c(p, \xi, S) = \varphi(p, v, S)\}.$$

The technical properties of the game φ are described in Lemma 1:

Lemma 1. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4),

- $(p, v) \to \varphi(p, v, S)$ *is continuous on* $\Delta \times \mathbb{R}_+^n$,
- $(p, v) \to \xi(p, v, S)$ *is an upper hemi-continuous correspondence on* $\mathring{\Delta} \times \mathbb{R}_+^n$.

The proof is straightforward and will be omitted.

The following proposition explains how the game $\varphi(p, v, \cdot)$ can be used to find allocations in the core of the economy.

PROPOSITION 2. *Under the assumptions* (A1)–(A3), (P1)–(P3), *if an allocation is blocked by a coalition* S, *then*

$$\forall p \in \mathring{\Delta}, \qquad \varphi(p, u(\xi), S) < p \cdot \sum_{i \in S} \omega_i.$$

The proof is very easy and left to the reader.

If there exists a Pareto optimal allocation ξ and a price $p \gg 0$ such that

$$\text{(I)} \quad p \cdot \sum_{i \in S} \omega_i \leqslant \varphi(p, u(\xi), S), \qquad \forall S \subset N,$$

this allocation is in the core of the economy since it is feasible and, from Proposition 2, no coalition can block it.

To find such an allocation, it is sufficient to find a Pareto imputation v and a price $p \gg 0$ such that the vector $(-p \cdot \omega_i)_{1 \leqslant i \leqslant n}$ belongs to the core of the game $-\varphi(p, v, \cdot)^2$.

The condition of increasing returns in production is sufficient to imply the existence of the core of $-\varphi(p, v, \cdot)$.

PROPOSITION 3. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4) *for all* $(p, v) \in \mathring{\Delta} \times \mathbb{R}^n_+$, *the game* $-\varphi(p, v, \cdot)$ *is balanced and therefore has a non-empty core.*

From a technical point of view, the introduction of a price p for the inputs and the study of the game $-\varphi(p, v, \cdot)$ which involves only cost functions enables us to use partial equilibrium methods. The proof of Proposition 3, given in Appendix, is thus very close of the proof given by Scarf [5] of the existence of the core of a two goods economy with increasing returns.

Now, if the vector $(-p \cdot \omega_i)_{1 \leqslant i \leqslant n}$ is in the core of a game $-\varphi(p, v, \cdot)$, for this couple (p, v) we will have the equality

$$\varphi(p, v, N) = \sum_{i=1}^{n} p \cdot \omega_i.$$

[2] We have to consider the core of the game $-\varphi(p, v, \cdot)$ because $\varphi(p, v, S)$ is not a gain but a cost for coalition S.

The condition of Pareto optimality of an imputation v implies that

$$\forall p \in \Delta, \qquad \varphi(p, v, N) \leqslant p \cdot \sum_{i=1}^{n} \omega_i,$$

since $U(v, N)$ contains at least one feasible allocation.

If for any p this inequality is strict there is no hope to prove that $(-p \cdot \omega_i)_{1 \leqslant i \leqslant n}$ belongs to the core of $-\varphi(p, v, \cdot)$.

Therefore, we will consider the condition

$$\text{(NE)} \quad \forall v \in P(u), \qquad \exists p \in \Delta, \qquad \varphi(p, v, N) = p \cdot \sum_{i=1}^{n} \omega_i.$$

One may be surprised that this condition is not always fulfilled, but the presence of increasing returns in production can arise this kind of phenomenon. It may be that, for all p, it is more advantageous, in term of cost, to choose a non-feasible allocation (with a higher production level) than a feasible one to achieve the Pareto imputation v.

The condition (NE) will be studied in the next section. We can prove that, under rather mild assumptions, this condition is sufficient to ensure the non-emptiness of the core of the economy.

PROPOSITION 4. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4) *if condition* (NE) *holds, there exists a Pareto imputation* v *and a price* $p \gg 0$ *such that* $(-p \cdot \omega_i)_{1 \leqslant i \leqslant n}$ *is in the core of the game* $-\varphi(p, v, \cdot)$.

The above reasoning shows that this proposition implies:

PROPOSITION 5. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4), *if condition* (NE) *holds, the core of the economy is non-empty.*

The proof of Proposition 4 requires the following lemma whose proof is given in the Appendix.

LEMMA 6. *Under the assumptions* (A1)–(A3), (P1)–(P3),

$$\forall v \in P(u), \qquad \left\{ p \;\middle|\; \varphi(p, v, N) = p \cdot \sum_{i=1}^{n} \omega_i \right\} \subset \mathring{\Delta}.$$

Proof of Proposition 4. Let A be the simplex of $\mathbb{R}^n$

$$A = \left\{ \alpha \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

The map θ

$$P(u) \to A$$

$$v \to \frac{v}{\sum_{i=1}^{n} v_i}$$

is continuous. Using an argument similar to Arrow and Hahn [1] and the monotonicity of preferences, one can prove that the inverse map θ^{-1} is well defined and continuous.

For $p \in \Delta$, $v \in P(u)$, let $-\mathcal{N}(p, v) = (-\mathcal{N}_i(p, v))_{1 \leqslant i \leqslant n}$ be the nucleolus of the game $-\varphi(p, v, \cdot)$. (For definition and properties see Schmeidler [8][3]). The continuity of φ implies that $(p, v) \to \mathcal{N}(p, v)$ is a continuous map on $\Delta \times \mathbb{R}^n_+$. When $p \in \mathring{\Delta}$, $-\varphi(p, v, \cdot)$ has a non-empty core and $-\mathcal{N}(p, v)$ belongs to this core.

Let ψ be the correspondence from $A \times \Delta$ into itself defined as follows:

$$(\alpha, p) \to \psi(\alpha, p) = (\delta(\alpha, p), \gamma(\alpha, p)),$$

where

$$\delta(\alpha, p) = \left\{ q \;\middle|\; \varphi(q, \theta^{-1}(\alpha), N) = q \cdot \sum_{i=1}^{n} \omega_i \right\},$$

$$\gamma(\alpha, p) = \{\beta \in A \mid \beta_i = 0 \text{ if } \mathcal{N}_i(p, \theta^{-1}(\alpha)) > p \cdot \omega_i\}.$$

Since condition (NE) holds $\delta(\alpha, p) \neq \varnothing$.

The continuity properties of θ^{-1}, φ and $\mathcal{N}$ imply that ψ is upper-hemi-continuous, and it is easy to verify that $\delta(\alpha, p)$ and $\gamma(\alpha, p)$ are convex valued. By application of Kakutani's fixed point theorem, the upper-hemi-continuous, convex valued correspondence ψ from $A \times \Delta$ into itself has a fixed point $(\bar{\alpha}, \bar{p})$.

Let us denote $\bar{v} = \theta^{-1}(\bar{\alpha})$.

Since $\bar{p} \in \delta(\bar{\alpha}, \bar{p})$, Lemma 6 implies that $\bar{p} \in \mathring{\Delta}$. Hence $-\mathcal{N}_i(\bar{v}, \bar{p})$ belongs to the core of $-\varphi(\bar{p}, \bar{v})$.

The proof will be complete if we prove that

$$\forall i, \qquad \bar{p} \cdot \omega_i = \mathcal{N}_i(\bar{p}, \bar{v}).$$

[3] The definition must be slightly modified since we consider a game with a negative caracteristic function. The nucleolus of the game $-\varphi(p, v, \cdot)$ is the nucleolus of the convex compact set

$$\left\{ (\alpha_i)_{1 \leqslant i \leqslant n} \;\middle|\; \sum_{i=1}^{n} \alpha_i = -\varphi(p, v, N),\ \alpha_i \leqslant 0 \right\}.$$

Suppose that there exists i_0 such that

$$\mathcal{N}_{i_0}(\bar{p}, \bar{v}) > \bar{p} \cdot \omega_{i_0}.$$

Then $\bar{\alpha}_{i_0} = 0$ and $\bar{v}_{i_0} = 0$.

This implies that $\varphi(p, v, \{i_0\}) = 0$ and

$$\forall S \subset N, \qquad \varphi(\bar{p}, \bar{v}, S \cup \{i_0\}) = \varphi(\bar{p}, \bar{v}, S).$$

Since $-\mathcal{N}(\bar{p}, \bar{v})$ belongs to the core of $-\varphi(\bar{p}, \bar{v}, \cdot)$ we have

$$-\mathcal{N}_{i_0}(\bar{p}, \bar{v}) \geqslant 0$$

and

$$-\sum_{i \neq i_0} \mathcal{N}_i(\bar{p}, \bar{v}) \geqslant \varphi(\bar{p}, \bar{v}, N - \{i_0\}) = \varphi(\bar{p}, \bar{v}, N) = -\sum_{i=1}^{n} \mathcal{N}_i(\bar{p}, \bar{v}).$$

These two inequalities imply that

$$\mathcal{N}_{i_0}(\bar{p}, \bar{v}) = 0$$

and then $\bar{p} \cdot \omega_{i_0} < 0$, which is impossible.

Hence $\forall i$, $\mathcal{N}_i(\bar{p}, \bar{v}) \leqslant \bar{p} \cdot \omega_i$.

As

$$\sum_{i=1}^{n} \mathcal{N}_i(\bar{p}, \bar{v}) = \varphi(\bar{p}, \bar{v}, N) = \bar{p} \cdot \sum_{i=1}^{n} \omega_i$$

we must have $\forall i$, $\mathcal{N}_i(\bar{p}, \bar{v}) = \bar{p} \cdot \omega_i$.

III. The Condition (NE)

Condition (NE) is always fulfilled in the simple case $l = 1$ (one input, one output). The price input p is then equal to 1 and for any allocation ξ, the cost condition $\varphi(1, \xi, N) \leqslant \sum_{i=1}^{n} \omega_i$ is equivalent to feasibility.

Therefore, for any Pareto imputation $v \in P(u)$, the minimum cost $\varphi(1, v, N)$ is exactly the sum of initial resources. One can remark too, that in the case $l = 1$, Δ equals $\mathring{\Delta}$ and the condition of strict monotonicity of preferences which is mainly needed to deal with boundary problems on the price simplex can be dispensed with. In this case, we get the same result than Scarf [5].

Theorem 7. *If $l = 1$, if preferences are continuous and monotonic, if the* *cost function is continuous, subhomogeneous and strictly increasing,*[4] *the core of the economy is non-empty.*

The purpose of this paper is mainly to explore the case $l > 1$ when all inputs enter as arguments of the utility functions, since no positive results are available in this case. Until now we have proved that, if Pareto optimal allocations are cost minimizing allocations for at least one input price system (condition NE), then the core of the economy is non-empty. But this condition is too far from being a condition on the caracteristics of the economy to be considered as really interesting.

In the following, we shall prove that this condition is implied by the unicity of the cost minimizing allocation (Proposition 8). Then we shall give sufficient conditions expressed in terms of the cost function and the compensated demand function which imply this unicity (Theorem 10).

Consider the following condition (Unicity of Cost Minimizing Allocation):

(UCMA) $\forall p \in \mathring{\Delta}$, $\forall v \in P(u)$, $\xi(p, v, N)$ has only one element.

PROPOSITION 8. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4), (C5), *condition* (UCMA) *implies condition* (NE).

Proposition 5 and 8 imply:

PROPOSITION 9. *Under the assumptions* (A1)–(A3), (P1)–(P3), (C4), (C5), *if condition* (UCMA) *holds, the core of the economy is non-empty.*

Proof of Proposition 8. Let $v \in P(u)$. We have to prove that there exists $p \in \Delta$ such that $\varphi(p, v, N) = p \cdot \sum_{i=1}^{n} \omega_i$.

For each $p \in \mathring{\Delta}$ denote

$$\xi(p) = (x_i(p), y_i(p))_{1 \leqslant i \leqslant n} = \xi(p, v, N),$$

$$y(p) = \sum_{i=1}^{n} y_i(p).$$

Condition (UCMA) and Lemma 1 imply that $p \to \xi(p)$ is a continuous function on $\mathring{\Delta}$.

To $y(p)$ associate $z(p) = -\zeta(p, y(p))$, the input combination which minimizes the cost of $y(p)$.

Then $\varphi(p, v, N) = p \cdot \sum_{i=1}^{n} x_i(p) + p \cdot z(p)$.

[4] This is equivalent to say that the production set can be described by a continuous increasing production function and presents non-decreasing returns to scale.

Since $v \in P(u)$ we have

$$p \cdot \sum_{i=1}^{n} x_i(p) + p \cdot z(p) - p \cdot \sum_{i=1}^{n} \omega_i \leqslant 0.$$

The function η defined by

$$\eta(p) = \sum_{i=1}^{n} x_i(p) + z(p) - \sum_{i=1}^{n} \omega_i$$

is continuous on $\mathring{\Delta}$ and plays the same role as the excess demand function in general equilibrium theory. We have

$$p \cdot \eta(p) \leqslant 0, \qquad \forall p \in \Delta, \tag{I}$$

and from the proof of Lemma 6, it is easy to infer that

$$\|\eta(p)\| \to \infty \qquad \text{when} \qquad p \to \Delta \backslash \mathring{\Delta} \tag{II}$$

Let us consider the correspondence Γ defined as follow: Denote $M(p) = \{h \in [1 \cdots l] \mid \eta_h(p) = \max_{1 \leqslant k \leqslant l} \eta_k(p)\}$. If $p \in \mathring{\Delta}$, $\Gamma(p) = \{q \mid q_n = 0$ if $h \notin M(p)\}$. If $p \in \Delta \backslash \mathring{\Delta}$, $\Gamma(p) = \{q \mid p \cdot q = 0\}$. $\Gamma(p)$ is convex valued and conditions (I) and (II) imply that Γ is an upper-hemi-continuous correspondence from Δ into itself. Then Γ has a fixed point $\bar{p}$. The definition of Γ implies that $\bar{p} \in \mathring{\Delta}$ and that all components of $\eta(\bar{p})$ are equal. The relation $\bar{p} \cdot \eta(\bar{p}) \leqslant 0$ implies that $\eta(\bar{p}) \leqslant 0$. The assumption (A2) of strict monotonicity of preferences and the fact that v is Pareto optimal imply that $\eta(\bar{p}) = 0$.

For the price $\bar{p}$ we have

$$\varphi(\bar{p}, v, N) = \bar{p} \cdot \eta(\bar{p}) + \bar{p} \cdot \sum_{i=1}^{n} \omega_i = \bar{p} \cdot \sum_{i=1}^{n} \omega_i.$$

Remark. When there are increasing returns, the program $\inf\{c(p, \xi, N) \mid \xi \in U(v, N)\}$ cannot be a convex program. Even if we take convexity assumptions on preferences (which imply the convexity of $U(v, N)$), the presence of increasing returns in production prevents the function $\xi \to c(p, \xi, N)$ from being convex or quasi-convex. Therefore the condition (UCMA) is necessary for the proof since generally, if $\xi(p, v, N)$ has more than one element, it is not convex valued and no fixed point theorem can be applied.

A geometric representation of the cost minimization problem in two dimensions gives the feeling that the unicity of $\xi(p, v, N)$ is determined by the respective curvatures of the production set and the Scitovsky indifference curves (Fig. 1). But it does not seem possible, in higher dimensions, to find a

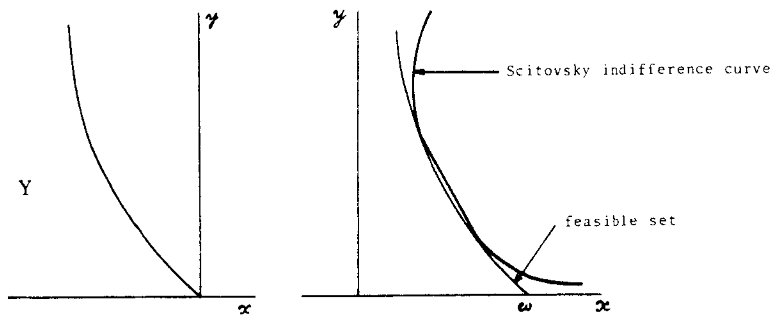

FIG. 1. A two goods economy.

second order condition which ensures the unicity of $\xi(p, v, N)$ and which bears directly on the primitive data of the economy—preference preorderings and production set. However, the logic of our study leads us to sufficient second order conditions expressed in terms of the cost function and of the (compensated) demand function.

More precisely, let us suppose that the preferences are strictly convex. If $v \geqslant 0$ is a utility level and $(p, q) \in \Delta \times \mathbb{R}^n_+$ a complete system of prices, the compensated demand of agent i, $M_i(p, q, v_i)$ is the unique vector $(x_i, y_i) \in \mathbb{R}^{l+1}_+$ which minimizes $p \cdot x_i + qy_i$ subject to $u_i(x_i, y_i) \geqslant v_i$.

If v is an imputation, the total compensated demand is

$$M(p, q, v) = \sum_{i=1}^{n} M_i(p, q, v_i).$$

The component of this total compensated demand in the produced good will be denoted $m(p, q, v)$.

The elasticity of the compensated demand of the output with respect to its price q will be denoted $\varepsilon(p, q, v)$.

$$\varepsilon(p, q, v) = \frac{\partial m/\partial q}{m}\, q.$$

The elasticity of the marginal cost of the output with respect to the level of production will be denoted $\eta(p, y)$.

$$\eta(p, y) = \frac{g''(p, y)}{g'(p, y)}\, y,$$

where $g'(p, y)$ and $g''(p, y)$ are the first and second derivatives of the cost function with respect to y.

With these notations we can state our main theorem.

THEOREM 10. *Under the assumptions* (A1)–(A5), (P1)–(P3), (C4)–(C6), *if one of the conditions* (I) *or* (II) *holds, then the core of the economy is non-empty.*

$$\text{(I)}\quad \forall p \in \mathring{\Delta} \;\; \forall v \in P(u) \qquad g''(p, y)\frac{\partial m}{\partial q}(p, q, v) < 1 \qquad \forall y \geqslant 0 \;\; \forall q > 0,$$

$$\text{(II)}\quad \forall p \in \mathring{\Delta} \;\; \forall v \in P(u) \qquad \varepsilon(p, y)\,\eta(p, q, v) < 1 \qquad \forall y \geqslant 0 \;\; \forall q > 0.$$

Before presenting the proof of Theorem 10, let us see what interpretation one can give to these conditions.

Condition (I) is familiar to many partial equilibrium analysis. When the price p is fixed for the inputs, and the levels of utility v are given, one can analyze the market of the output: $q \to m(p, q, v)$ is the (compensated) demand function and $y \to g(p, y)$ is the cost function.

Condition (I) can be written as

$$g''(p, y) > \frac{1}{\partial m/\partial q} \qquad \left(\text{since } \frac{\partial m}{\partial q} < 0\right).$$

This is an inequality between the slopes of the marginal cost curve and the inverse demand curve. In particular, this inequality implies that the demand curve intersects the marginal cost curve from above.

Since Condition (I) consists of a lower bound on the derivative of the marginal cost function, it will be satisfied if there is no level of production where the marginal cost is very quickly decreasing.

Once can imagine easily a case where this condition would be violated. Let us suppose that there exists two techniques of production for the output: the first one requires a lower amount of investment than the second but leads to a higher marginal cost. For small quantities of output, it is cheaper to use the first technique. For some level of production (which may depend on the input price), it becomes profitable to change for the second one. At this point,

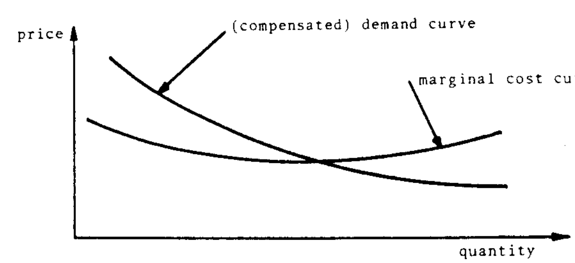

FIG. 2. Market of the output: p and v are fixed.

there is an abrupt decrease in the marginal cost. Of course, to respect the assumptions of the model, one has to smooth the description of production possibilities but this is always possible by approximation.

Let us note that the smaller is $|\partial m/\partial q|$, the lower is the bound on $g''(p, y)$, and the larger is the set of technologies which will lead to a non-empty core. If one adopts the traditional view that a low elasticity in demand happens when the output has no close substitutes in the economy, it is not surprising that this fact enforces the stability of collective production, to take advantage of the economies of scale.

Condition (II) has essentially the same interpretation than condition (I). The main interest of setting the inequality in terms of elasticities is to make the proof of Corollary II easier.

Finally, let us remark that both conditions (I) and (II) hold with constant returns to scale since, in this case, $g''(p, y) = 0$. Theorem 10 shows that the property of existence of the core when there are constant returns to scale holds true by continuity when returns to scale are increasing, as long as the quantity $g''(p, y)(\partial m/\partial q)$ stays bounded by 1.

Proof of Theorem 10. Let $p \subset \mathring{\Delta}$, $v \in P(u)$.

$\xi(p, v, N)$ is the set of solutions of the program:

$$\inf p \cdot \sum_{i=1}^{n} x_i + g(p, y)$$

subject to:

$$\cdot\ u_i(x_i, y_i) \geqslant v_i, \qquad \forall i \in [1 \cdots n],$$

$$\cdot\ \sum_{i=1}^{n} y_i = y,$$

$$\cdot\ x_i \geqslant 0,\ y_i \geqslant 0, \qquad \forall i \in [1 \cdots n].$$

It is clear that, for any solution, $(x_i, y_i) = (0, 0)$ when $v_i = 0$. Therefore, we can consider only the agents i such that $v_i > 0$, or assume, w.l.o.g., that $\forall i \in [1 \cdots n]\ v_i > 0$. Then the assumptions on the preferences imply that the positivity constraints are not tight.

The qualification condition on the constraints is clearly satisfied and all the solutions of the program must satisfy the first order conditions. We shall prove the unicity of the solution of the program by proving that the first order conditions cannot have more than one solution.

These conditions are:

$$\exists(\lambda_i)_{1 \leqslant i \leqslant n} \qquad \lambda_i \geqslant 0 \qquad \exists \mu \geqslant 0 \qquad \exists (x_i, y_i)_{1 \leqslant i \leqslant n}$$

such that:

$$p_h = \lambda_i \frac{\partial u_i}{\partial x_{ih}}, \qquad \forall h \in [1 \cdots l] \quad \forall i \in [1 \cdots n], \tag{1}$$

$$\mu = \lambda_i \frac{\partial u_i}{\partial y_i}, \qquad \forall i \in [1 \cdots n], \tag{2}$$

$$u_i(x_i, y_i) = v_i, \tag{3}$$

$$g'(p, y) = \mu, \tag{4}$$

$$\sum_{i=1}^{n} y_i = y. \tag{5}$$

Strict convexity of u_i implies that Eqs. (1), (2), (3) are equivalent to:

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = M_i(p, \mu, v_i), \qquad \forall i \in [1 \cdots n].$$

μ is determined with y by the equations:

$$\begin{cases} (4) & g'(p, y) = \mu \\ (6) & m(p, \mu, v) = y \end{cases} \Leftrightarrow \begin{cases} g'(p, y) = \mu & (4) \\ m(p, g'(p, y), v) = y. & (7) \end{cases}$$

If $(\partial m/\partial q)(p, q, v)\, g''(p, y) < 1$, $\forall y > 0$, $\forall q > 0$, the function $y \to m(p, g'(p, y), v) - y$ is decreasing and Eq. (7) cannot have more than one solution.

The assumption (A2) on preferences imply that $y > 0$ so that condition (7) can be written:

$$\frac{m(p, g'(p, y), v)}{y} = 1 \tag{8}$$

if $\varepsilon(p, q, v)\, \eta(p, y) < 1$, $\forall y > 0$, $\forall q > 0$,

$$\frac{d}{dy}\left(\frac{m(p, g'(p, y), v)}{y}\right) < 0$$

and Eq. (8) cannot have more than one solution.

The Theorem 10 enlarges the class of the economies for which one can prove that the core is non-empty, as shown by the following corollary:

COROLLARY 11. *If the economy is such that*

— *the utility functions are of Cobb–Douglas type,*

— *the production set is described either by a Cobb–Douglas or by a C.E.S. production function with increasing returns,. then the core of the economy is non-empty.*

Proof. It is easy to check that in all cases:

$$-1 < \varepsilon(p, q, v) < 0, \qquad \forall p \in \mathring{\Delta}, \ \forall q > 0, \ \forall v \in P(u),$$

$$-1 < \eta(p, y) < 0, \qquad \forall p \in \mathring{\Delta}, \ \forall y > 0.$$

and condition (II) holds.

APPENDIX

Proof of Proposition 3. Let $T = \{S_k \mid 1 \leqslant k \leqslant m\}$ be a balanced family of coalitions with balancing weights $(\delta_k)_{1 \leqslant k \leqslant m}$.

We have to prove that

$$\sum_{k=1}^{m} \delta_k \varphi(p, v, S_k) \geqslant \varphi(p, v, N), \qquad \forall p \gg 0, \quad \forall v \in \mathbb{R}^n_+. \tag{I}$$

$p \gg 0$ and $v \in \mathbb{R}^n_+$ having been chosen, for each $k \in [1 \cdots m]$, let $\xi^k = (x^k_i, y^k_i)_{1 \leqslant i \leqslant n}$ be an allocation in $\xi(p, v, S_k)$.

Denote

$$a(S_k) = \frac{g(p, \sum_{i \in S_k} y^k_i)}{\sum_{i \in S_k} y^k_i}.$$

As in Scarf [5, p. 229], there is no loss of generality in assuming that the indices have been chosen such that $a(S_1) \leqslant a(S_2) \leqslant \cdots \leqslant a(S_m)$, and that there exists an index k_0 such that $\delta_{S_1} + \cdots + \delta_{S_{k_0}} = 1$.

— if $k_0 = m$, $\forall k \; S_k = N$ and the inequality (I) is trivially true;

— if $k_0 < m$ denote

$$A = \{S_k \mid 1 \leqslant k \leqslant k_0\},$$

$$B = \{S_k \mid k_0 + 1 \leqslant k \leqslant m\}.$$

For each coalition $S_k \in A$, denote $U^k(v, N)$ the set of the allocations ξ which coincide with ξ^k on S_k and give to each agent $i \notin S_k$ at least the utility level v_i

$$U^k(v, N) = \left\{ \xi \in \mathbb{R}^{(l+1)n} \;\middle|\; \begin{array}{l} \forall i \in S_k, \xi_i = \xi^k_i = (x^k_i, y^k_i) \\ \forall i \notin S_k, u_i(\xi_i) = u_i(x_i, y_i) \geqslant v_i \end{array} \right\}.$$

If $\xi \in U^k(v, N)$, then $\xi \in U(v, N)$ and

$$\varphi(p, v, N) \leqslant c(p, \xi, N) = p\left(\sum_{i \in S_k} x^k_i + \sum_{i \notin S_k} x_i\right) + g\left(p, \sum_{i \in S_k} y^k_i + \sum_{i \notin S_k} y_i\right).$$

The assumption of increasing returns imply that $y \to g(p, y)/y$ is a decreasing function of y. We have then the inequality

$$\varphi(p, v, N) \leqslant p\left(\sum_{i \in S_k} x^k_i + \sum_{i \notin S_k} x_i\right) + a(S_k)\left(\sum_{i \in S_k} y^k_i + \sum_{i \notin S_k} y_i\right). \tag{II}$$

Let us consider the random allocation built in the following way: we select at random a coalition S_k in A with probability $\delta_k (\sum_{S_k \in A} \delta_k = 1)$. S_k been selected, to each agent $i \in S_k$ we give $\xi^k_i = (x^k_i, y^k_i)$. Then we select, independently for each $i \notin S_k$, a coalition S_l from those coalitions of B that contain i, with conditional probability $\delta_l / \sum_{S_{l'} \in B, i \in S_{l'}} \delta_{l'}$.

When S_l is selected, we give $\xi_i = (x^l_i, y^l_i)$ to agent i. Each realization of the random variable ξ verifies inequality (II). Therefore $\varphi(p, v, N)$ is smaller than the expectation of the right-hand side of inequality (II).

We get:

$$\varphi(p, v, N) \leqslant \sum_{S_k \in A} \delta_k \Bigg[p \cdot \sum_{i \in S_k} x^k_i + a(S_k) \sum_{i \in S_k} y^k_i + \sum_{i \notin S_k} \sum_{\substack{S_l \in B \\ i \in S_l}} \frac{\delta_l}{\sum_{S_{l'} \in B, i \in S_{l'}} \delta_{l'}} (p \cdot x^l_i + a(S_k)\, y^l_i) \Bigg].$$

We can write this inequality as

$$\varphi(p, v, N) \leqslant \sum_{S_k \in A} \delta_k \varphi(p, v, S_k) + \sum_{S_l \in B} \delta_l \sum_{i \in S_l} \frac{\sum_{S_k \in A, i \notin S_k} \delta_k}{\sum_{S_{l'} \in B, i \in S_{l'}} \delta_{l'}} (p \cdot x^l_i + a(S_k)\, y^l_i).$$

But let us remark that

· $\forall S_k \in A, \qquad \forall S_l \in B, \qquad a(S_k) \leqslant a(S_l),$

and

· $\forall i \in N, \qquad$ the ratio $\dfrac{\sum_{S_k \in A, i \notin S_k} \delta_k}{\sum_{S_{l'} \in B, i \in S_{l'}} \delta_{l'}}$ is equal to 1 since

$$\sum_{\substack{S_k \in A \\ i \notin S_k}} \delta_k = 1 - \sum_{\substack{S_k \in A \\ i \in S_k}} \delta_k = 1 - \left(1 - \sum_{\substack{S_{l'} \in B \\ i \in S_{l'}}} \delta_{l'}\right).$$

Hence we have

$$\varphi(p, v, N) \leqslant \sum_{S_k \in A} \delta_k \varphi(p, v, S_k) + \sum_{S_l \in B} \delta_l \left(\sum_{i \in S_l} p \cdot x_i^l + a(S_l)\, y_i^l\right),$$

which is inequality (I).

Proof of Lemma 6. Let $v \in P(u)$. There exists i_0 such that $v_{i_0} > 0$. Let us suppose that there exists $\bar{p} \in \Delta \backslash \mathring{\Delta}$ such that

$$\varphi(\bar{p}, v, N) = \bar{p} \cdot \sum_{i=1}^{n} \omega_i.$$

Let h be a component such that $p_h \neq 0$. Let k be a component such that $p_k = 0$.

Since v is Pareto optimal, there exists a feasible allocation $\bar{\xi}$ such that $u_i(\bar{\xi}_i) = v_i$, $\forall i \in \{1, \ldots, n\}$.

From (A3) (boundary condition), $\bar{\xi}$ is such that $\bar{\xi}_{i_0} \in \mathbb{R}_{++}^{l+1}$.

From (A1) (strict monotonicity) we have

$$\forall \varepsilon > 0, \qquad u_{i_0}(\bar{\xi}_{i_0} + \varepsilon l_k) > v_{i_0},$$

where l_k denotes the kth unit vector of $\mathbb{R}^{l+1}$.

From (A1) (continuity) and $x_{i_0 h} > 0$, we can deduce that

$$\exists \alpha > 0, \qquad u_{i_0}(\bar{\xi}_{i_0} + \varepsilon l_k - \alpha l_h) \geqslant v_{i_0}.$$

Denote ξ' the allocation such that

$$\xi_i' = \bar{\xi}_i, \qquad \forall i \neq i_0,$$
$$\xi_{i_0}' = \bar{\xi}_{i_0} + \varepsilon l_k - \alpha l_h.$$

Since ξ' belongs to $U(v, N)$ we have

$$\varphi(\bar{p}, v, N) \leqslant C(\bar{p}, \xi', N) < C(\bar{p}, \xi, N) \leqslant \bar{p} \cdot \sum_{i=1}^{n} \omega_i,$$

which contradicts the equality

$$\varphi(\bar{p}, v, N) = \bar{p} \cdot \sum_{i=1}^{n} \omega_i.$$

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